On a class of boundary value problems involving the p-biharmonic operator

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\textbf{A B S T R A C T}

A nonlinear boundary value problem involving the p-biharmonic operator is investigated, where $p > 1$. It describes various problems in the theory of elasticity, e.g., the shape of an elastic beam where the bending moment depends on the curvature as a power function with exponent $p - 1$. We prove existence of solutions satisfying a quite general boundary condition that incorporates many particular boundary conditions which are frequently considered in the literature.

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\section{1. Introduction}

It is the purpose of this paper to investigate the following nonlinear, nonsmooth, fourth order boundary value problem

$$
\left( |u''|^p - 2 u'' \right)'' - (a(t)|u'|^{p-2}u')' + b(t)|u|^{p-2}u \in \delta F(t, u),
$$

(1)

$$
\left( |u''|^p - 2 u''(0) + a(0)|u'(0)|^{p-2}u'(0)
\begin{pmatrix}
|u''(0)|^{p-2}u''(0) \\
-|u''(1)|^{p-2}u''(1)
\end{pmatrix}
\in \partial j
\begin{pmatrix}
|u(0)| \\
|u'(0)| \\
|u(1)| \\
|u'(1)|
\end{pmatrix},
$$

(2)

where $a, b \in C^0([0, 1])$ are given real functions, $p > 1$ and $F, j$ are nonlinear functions satisfying some conditions which are specified below. Both Eq. (1) and the boundary condition (2) are sufficiently general to cover a broad range of specific problems. Our treatment is mainly based on a variational approach.

To be more specific, let us formulate our assumptions on $F$ and $j$:

\begin{itemize}
  \item[$(H_1)$] $F = F(t, \xi): (0, 1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, satisfying in addition $F(t, 0) = 0$ for a.a. $t \in (0, 1)$, as well as the Lipschitz condition:
  \[ \forall \rho > 0 \text{ there is an } \alpha_\rho \in L^1(0, 1) \text{ such that } \]
  \[ |F(t, x) - F(t, y)| \leq \alpha_\rho(t)|x - y|, \]
  \[ \text{for a.a. } t \in (0, 1) \text{ and all } x, y \text{ with } |x|, |y| \leq \rho; \]
\end{itemize}

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Function $j : \mathbb{R}^4 \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous (l.s.c.), such that the null (column) vector $(0, 0, 0, 0)^T \in D(j)$.

Now, to complete the presentation of our boundary value problem, let us explain the notation we have used above in (1) and (2). $\partial F(t, \xi)$ denotes the generalized Clarke gradient of $F(t, \cdot)$ at $\xi \in \mathbb{R}$, while $\partial j$ stands for the subdifferential of $j$ (see [14]).

Note that our conditions $F(t, 0) = 0$ and $(0, 0, 0, 0)^T \in D(j)$ do not restrict much the generality of the problem. In fact, the later can always be reached by a translation of $u$. Of course, this operation changes Eq. (1), but the treatment is similar. In particular, if either $p = 2$ or $b$ is the null function, Eq. (1) remains unchanged and it suffices to assume that $F(t, 0)$ is an $L^1$ function.

A classical fourth order equation arising in the beam-column theory is the following (see Timoshenko and Gere [15])

$$EI \frac{d^4 u}{dx^4} + p \frac{d^2 u}{dx^2} = q,$$

(3)

where $u$ is the lateral deflection, $q$ is the intensity of a distributed lateral load, $P$ is the axial compressive force applied to the beam and $EI$ represents the flexural rigidity in the plane of bending. Eq. (3) is derived from the static equilibrium equations for any slice at distance $x$ along the beam, namely the equilibrium of forces reads

$$q = -\frac{dV}{dx},$$

(4)

where $V$ is the shearing force and the equilibrium of moments is expressed by the equation

$$V = \frac{dM}{dx} - p \frac{du}{dx},$$

(5)

where $M$ denotes the bending moment. It is assumed that the bending moment depends linearly on the curvature. It can be expressed (if some higher order terms are neglected) as follows

$$EI \frac{d^2 u}{dx^2} = -M.$$  

(6)

Let us consider a more general situation, that the bending moment is a power function of the curvature with exponent $p - 1$, i.e.,

$$M = -c \left| \frac{d^2 u}{dx^2} \right|^{p-2} \frac{d^2 u}{dx^2},$$

(7)

where $c$ is a constant. Then the presence of the term $(u''(p-2)u'')''$ in (1) is justified if we assume (7) instead of (6) when Eq. (3) is derived. If $p = 2$, then (7) coincides with (6) where $c = EI$.

Another equation that motivates our investigation here is the following one

$$Dw'' + Nw'' + Eh \frac{W}{a^2} = q, \quad t \in (0, 1).$$

(8)

It models the radial deflection $w$ for symmetrical buckling of a cylindrical shell under uniform axial compression $N_x$ (see [15, p. 457], [13]).

The applied lateral load $q$ in (3) or (8) represents the reaction of a support, which generally depends nonlinearly on the deflection (see [3–8,12,13])

$$q(t) = f(t, u(t)),$$

or, more generally,

$$q(t) \in \partial F(t, u(t)),$$

where $F$ is a nonsmooth function (in particular, $F$ may have some jumps, e.g., the case of adhesive support, see [13]).

Condition (2) covers many different types of boundary conditions (see [9]). For example, it is easy to check that for

$$j((x_1, x_2, x_3, x_4)^T) := \begin{cases} 0, & x_1 = x_2, \ x_3 = x_4, \\ +\infty, & \text{otherwise}. \end{cases}$$

we obtain the periodic conditions $u^{(i)}(0) = u^{(i)}(1), i = 0, 1, 2, 3$, while the case of simply supported endpoints, i.e., $u(0) = u(1) = u''(0) = u''(1) = 0$, corresponds to the following choice

$$j((x_1, x_2, x_3, x_4)^T) := \begin{cases} 0, & x_1 = x_2 = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

We encourage the reader to find $j$ for other types of classical boundary conditions (in particular for $p = 2$ and $a, b$ constants).
Our aim in this paper is to investigate the existence of solutions to problem (1)–(2). We extend our previous results related to the particular case $p = 2$ and $a$, $b$ constants [6]. The treatment of the more general problem (1), (2) requires more advanced analysis.

By a solution of this problem we mean a function $u \in W^{2,p}(0,1)$, with $|u''|^p - u'' \in AC([0,1],\mathbb{R})$, which satisfies (2) and for a.a. $t \in (0,1)$

\[
\left(|u''(t)|^p - u''(t)\right)'' - (a(t)|u'(t)|^{p-2}u'(t))' + b(t)|u(t)|^{p-2}u(t) \in \tilde{F}(t,u).
\]

(9)

Here, $AC([0,1],\mathbb{R})$ denotes the space of all absolutely continuous real functions defined on $[0,1]$. In fact, since $|u''|^p - u'' = v \in W^{2,1}(0,1)$, and $u'' = |v|^q v$, where $q$ is the conjugate of $p$ (i.e., $p^{-1} + q^{-1} = 1$), it follows that $u \in C^2([0,1])$. In particular, the values of $u$, $u'$, $u''$ at $t = 0$ and $t = 1$ in (2) make sense. Note that if $1 < p \leq 2$ then $u \in C^3([0,1])$.

Now, we define the set

\[ D = \{ u: u \in W^{2,p}(0,1), (u(0), u(1), u'(0), u'(1))^T \in D(j) \}, \]

and the functional $J : W^{2,p}(0,1) \to \mathbb{R} \cup \{ +\infty \},$

\[ J(u) := j((u(0), u(1), u'(0), u'(1))^T), \quad \forall u \in W^{2,p}(0,1), \]

whose effective domain is $D(j) = D,$

Obviously, $\mathcal{D} \neq \emptyset$ since $(0, 0, 0, 0)^T \in D(j)$, so $J$ is proper, convex and l.s.c.

In order to obtain existence of solutions to problem (1), (2), we consider the following functional

\[ I(u) := \frac{1}{p} \int_0^1 \left( |u''|^p + a|u'|^p + b|u|^p \right) dt - \int_0^1 F(t, u) dt + J(u), \]

and use a technique similar to that developed in Motreanu and Panagiotopoulos [13].

Let us define the following two constants,

\[ \lambda_1 := \inf \left\{ \frac{\int_0^1 \left( |u''|^p + a|u'|^p + b|u|^p \right) dt}{\|u\|_{L^p}^p}: u \in \mathcal{D}\setminus\{0\} \right\}, \]

and

\[ \tilde{\lambda}_1 := \lim_{s \to \infty} \inf_{ru \in \mathcal{D}} \left\{ \frac{\int_0^1 \left( |u''|^p + a|u'|^p + b|u|^p \right) dt + \frac{p J(ru)}{r^p}}{\|u\|_{L^p}^p} : \|u\|_{L^p}^p = 1 \right\}. \]

(11)

Both $\lambda_1$ and $\tilde{\lambda}_1$ will be important in the sequel. It is easily seen that $\lambda_1 \leq \tilde{\lambda}_1$, but in most cases $\lambda_1 < \tilde{\lambda}_1$.

We are now able to state the main results of the present paper, as follows.

**Theorem 1.1.** Assume $(H_1)$ and $(H_2)$. Suppose, in addition, that the following condition is satisfied

\[
(L_1) \quad \limsup_{|x| \to \infty} \frac{F(t, x)}{|x|^p} < \frac{\lambda_1}{p},
\]

uniformly for a.a. $t \in (0, 1).$ Then problem (1), (2) has at least a solution.

In order to state our next result, we introduce a new condition on $F$:

\[
(L_0) \quad \limsup_{x \to 0} \frac{F(t, x)}{|x|^p} < \frac{\lambda_1}{p},
\]

uniformly for a.e. $t \in (0, 1).$ Obviously this implies $0 \in \tilde{F}(t, 0)$ for a.a. $t \in (0, 1)$, so in this case $u(t) \equiv 0$ is a solution of problem (1), (2). We are interested in the existence of nontrivial solutions of problem (1), (2). We have

**Theorem 1.2.** Assume that $\lambda_1 > 0$ and that $(L_0)$, $(H_1)$, and $(H_2)$ are fulfilled. Suppose, in addition, that $D(j)$ is closed, $(0, 0, 0, 0)^T \in \partial j((0, 0, 0, 0)^T)$, and either $(G_o)$ or $(G_p)$–$(L_\infty)$ holds, where

$(G_o)$ there exist constants $\theta > p,$ and $k, M > 0,$ such that

\[ j'(x; z) \leq \theta j(z) + k, \quad \forall z \in D(j), \]

\[ 0 < \theta F(t, x) \leq \xi x, \quad \forall x \in \tilde{F}(t, x), \]

for all $|x| > M$, and a.a. $t \in (0, 1),$. 

there exist positive constants \(c, k, M\) such that
\[
j'(z; z) \leq pj(z) + k, \quad \forall z \in D(j),
\]
\[
0 < \left( p + \frac{c}{|x|^{p-1}} \right) F(t, x) \leq \xi x, \quad \forall x \in \partial F(t, x),
\]
for all \(|x| > M\), and a.a. \(t \in (0, 1)\), and
\[
(\liminf_{|x| \to \infty}) \frac{F(t, x)}{|x|^p} > \frac{\bar{\lambda}_1}{p},
\]
uniformly for a.a. \(t \in (0, 1)\).

Then problem (1) has at least a nonzero solution.

**Remark 1.1.** It is worth pointing out that any of the conditions (13) and (15) implies that the domain \(D(j)\) of functional \(j\) is a convex cone. Moreover, assumption (15) guarantees that \(\bar{\lambda}_1 < \infty\) (see Lemma 2.2 below).

## 2. Preliminaries

Let \(X\) be a Banach space whose dual is denoted by \(X^*\). We recall that the generalized directional derivative \(\Phi^0(u; v)\) of a locally Lipschitz function \(\Phi : X \to \mathbb{R}\) at a point \(u \in X\) and in the direction \(v \in X\) is defined by
\[
\Phi^0(u; v) := \limsup_{w \to u, s \to 0} \frac{\Phi(w + sv) - \Phi(w)}{s}.
\]
The set
\[
\partial \Phi(u) := \{ \eta \in X^* : \Phi^0(u; v) \geq (\eta, v), \forall v \in X \}
\]
denotes the generalized gradient \(\partial \Phi(u)\) of the function \(\Phi\) (in the sense of Clarke [1,2]).

Assume that the functional \(I\) has the form
\[
I = \Phi + \psi,
\]
where \(\Phi : X \to \mathbb{R}\) is a locally Lipschitz function, and \(\psi : X \to (-\infty, +\infty]\) is a proper, convex and l.s.c. function. We recall the definition of a critical point of the functional \(I\) as well as the Palais–Smale condition.

**Definition 2.1.** A vector \(u \in X\) is said to be a critical point of the functional \(I = \Phi + \psi\) if the following inequality holds
\[
\Phi^0(u; v - u) + \psi(v) - \psi(u) \geq 0, \quad \forall v \in X.
\]
A number \(c \in \mathbb{R}\) such that \(I^{-1}(c)\) contains a critical point is called a critical value of \(I\).

**Definition 2.2.** The functional \(I\) is said to satisfy the Palais–Smale (PS) condition if every sequence \(\{u_n\} \subset X\) such that \(|I(u_n)| < C\) with a constant \(C\) and
\[
\Phi^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,
\]
for a sequence \(\{\varepsilon_n\} \subset \mathbb{R}^+\) with \(\varepsilon_n \to 0\), possesses a convergent subsequence.

In what follows we need the following generalized mountain pass theorem (cf. [13]; see also [11] and [16]).

**Theorem 2.1 (Mountain pass).** Suppose that \(I\) satisfies the (PS) condition, \(I(0) = 0\) and

(i) there exist \(\alpha, \rho > 0\) such that \(I(u) \geq \alpha\) if \(|u| = \rho\),

(ii) \(I(e) \leq 0\) for some \(e \in X\), with \(|e| > \rho\).

Then the number
\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I(\gamma(t)),
\]
where
\[
\Gamma = \{ \gamma \in \mathcal{C}([0, 1], X) : \gamma(0) = 0, \gamma(1) = e \}
\]
is a critical value of \(I\) with \(c \geq \alpha\).
Our framework in the next part of the paper will be $X = W^{2,p}(0, 1)$.

Define $\psi : X \to (-\infty, +\infty]$ by

$$
\psi(u) := \frac{1}{p} \int_{0}^{1} \left( |u''|^p + |u'|^p + |u|^p \right) dt + J(u)
= \frac{1}{p} \|u\|^p_{W^{2,p}(0,1)} + J(u),
$$

and

$$
\varphi(u) := \frac{1}{p} \int_{0}^{1} \left( (a(t) - 1)|u'(t)|^p + (b(t) - 1)|u(t)|^p \right) dt, \quad u \in X.
$$

Notice that $\psi$ is a proper, convex and l.s.c. functional whose effective domain is $D(\psi) = D$, while $\varphi \in C^{1}(W^{2,p}(0, 1), \mathbb{R})$, and

$$
\langle \varphi'(u), v \rangle = \int_{0}^{1} \left( (a(t) - 1)|u'(t)|^{p-2}u'(t)v'(t) + (b(t) - 1)|u(t)|^{p-2}u(t)v(t) \right) dt.
$$

The following proposition characterizes the generalized gradient $\partial\Phi(u)$ of the functional

$$
\Phi(u) := -\frac{1}{p} \int_{0}^{1} F(t, u) dt + \varphi(u), \quad u \in X = W^{2,p}(0, 1).
$$

**Proposition 2.1.** Assume that $F : (0, 1) \times \mathbb{R} \to \mathbb{R}$ satisfies (H1). Then the functional $\Phi$ defined by (19) is locally Lipschitz. Moreover, if $u \in W^{2,p}(0, 1)$ and $l \in \partial\Phi(u)$ then there is some $u_l \in L^1(0, 1)$ such that $u_l(t) \in \partial F(t, u(t))$ for a.a. $t \in (0, 1)$, and

$$
\langle l, v \rangle = \int_{0}^{1} \left( -u_l(t)v(t) + (a(t) - 1)|u'(t)|^{p-2}u'(t)v'(t) \right.
+ (b(t) - 1)|u(t)|^{p-2}u(t)v(t) \left. \right) dt, \quad \forall v \in W^{2,p}(0, 1), \tag{20}
$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $W^{2,p}(0, 1)$.

**Proof.** By the continuity of the embedding $W^{2,p}(0, 1) \subset C^1([0,1])$ and assumption (H1) it follows that $\Phi$ is indeed locally Lipschitz.

Define

$$
R(u) := -\int_{0}^{1} F(t, u(t)) dt, \quad \forall u \in W^{2,p}(0, 1).
$$

One can prove (see Theorem 2.7.3 in [2]) that given $\xi \in \partial R(u)$ there then exists some $v_\xi \in L^1(0, 1)$ such that $v_\xi(t) \in -\partial F(t, u(t))$ for a.a. $t \in (0, 1)$, and

$$
\langle \xi, v \rangle = \int_{0}^{1} v_\xi v dt, \quad \forall v \in W^{2,p}(0, 1). \tag{21}
$$

For a complete direct proof of this assertion we refer the reader to [6, p. 2804].

Now, let $l \in \partial \Phi(u)$. By $\partial \Phi(u) \subset \partial R(u) + \partial \varphi(u) = \partial R(u) + \varphi'(u)$, there exists $\xi \in \partial R(u)$ such that $l = \xi + \varphi'(u)$ and (20) is obtained with $u_l := -v_\xi$, where $v_\xi$ is determined by $\xi$ as above. \(\square\)

Define $I : X = W^{2,p}(0, 1) \to (-\infty, +\infty]$ by

$$
I(u) := \Phi(u) + \psi(u) = \frac{1}{p} \int_{0}^{1} \left( |u''|^p + a|u'|^p + b|u|^p \right) dt - \int_{0}^{1} F(t, u) dt + J(u).
$$
Theorem 2.2. If $F: (0, 1) \times \mathbb{R} \to \mathbb{R}$ satisfies (H1) and $u \in W^{2,p}(0, 1)$ is a critical point of functional $I$, then $u$ is a solution of problem (1), (2).

Proof. We adapt a previous device from [10, Proposition 3.2], to the present functional (see also [6, p. 2805]). If we take $v = u + sw$, $s > 0$, in the inequality

$$\phi^0(u; v - u) + \psi(v) - \psi(u) \geq 0, \quad \forall v \in W^{2,p}(0, 1),$$

we easily get

$$\phi^0(u; w) + \int_0^1 \left( |u''|^p - 2u''w'' + |u'|^p - 2u'w' + |u|^p - 2uw \right) dt + J'(u; w) \geq 0, \quad (22)$$

for all $w \in W^{2,p}(0, 1)$, where

$$J'(u; w) = J'((u(0), u(1), u'(0), u'(1))^T; (w(0), w(1), w'(0), w'(1))^T) \quad (23)$$

is the directional derivative of the convex function $J$ at $u$ in the direction $w$. For $w \in C^\infty_0(0, 1) \subset W^{2,p}(0, 1)$, inequality (22) reads

$$\phi^0(u; w) \geq - \int_0^1 \left( |u''|^p - 2u''w'' + |u'|^p - 2u'w' + |u|^p - 2uw \right) dt. \quad (24)$$

The Hahn–Banach theorem implies that there exists a linear functional $I: W^{2,p}(0, 1) \to \mathbb{R}$, such that

$$\phi^0(u; w) \geq I(w), \quad \forall w \in W^{2,p}(0, 1), \quad (25)$$

and

$$I(w) = - \int_0^1 \left( |u''|^p - 2u''w'' + |u'|^p - 2u'w' + |u|^p - 2uw \right) dt, \quad \forall w \in C^\infty_0(0, 1)$$

as far as the function $\phi^0(u; \cdot)$ is subadditive and positively homogeneous. Moreover, the estimate

$$\phi^0(u; w) \leq k\|w\|_{W^{2,p}(0, 1)}, \quad \forall w \in W^{2,p}(0, 1) \quad (26)$$

holds with $k > 0$ being a Lipschitz constant of $\phi$ in a vicinity of $u$. Hence,

$$|I(w)| \leq k\|w\|_{W^{2,p}(0, 1)}, \quad \forall w \in W^{2,p}(0, 1),$$

showing that $I$ is continuous. The inequality (25) yields that $l \in \partial \phi(u)$. Now, there is some $u_1 \in L^1(0, 1)$ such that

$$u_1(t) \in \partial F(t, u(t)), \quad \text{for a.a. } t \in (0, 1), \quad (27)$$

and

$$\int_0^1 \left( |u''|^p - 2u''w'' + a|u'|^p - 2u'w' + b|u|^p - 2uw - u_1w \right) dt = 0, \quad (28)$$

for all $w \in C^\infty_0(0, 1)$ as a consequence of Proposition 2.1. Since $u \in W^{2,p}(0, 1)$, we have $(|u''|^p - 2u'')' \in W^{1,1}(0, 1)$, i.e., $(|u''|^p - 2u'')'$ is absolutely continuous and

$$\left( |u''(t)|^p - 2u''(t) \right) + \left( a(t)|u'(t)|^p - 2u'(t) \right) + b(t)|u(t)|^p - 2u(t) = u_1(t) \quad (29)$$

for a.a. $t \in (0, 1)$. Then (9) easily follows from (27).

Next, we prove that $u$ satisfies (2). We already know that $u'' \in C^2([0, 1])$. The above inclusion relation (27) implies that

$$u_1(t)w(t) \leq F^0(t, u(t); w(t)) \quad \text{for a.a. } t \in (0, 1), \quad \forall w \in W^{2,p}(0, 1).$$

Then, by (29), we have
\[
\begin{align*}
\int_0^1 \left( |u''|^2 u'' + a|u'|^2 u' + b|u|^2 u \right) dt + \left( |u''|^2 u'' \right) (1) - a(1)|u'(1)|^2 u(1) w(1) \\
- \left( |u''|^2 u'' \right) (0) - a(0)|u'(0)|^2 u'(0) w(0) - |u''(1)|^2 u''(1) w'(1) + |u''(0)|^2 u''(0) w'(0) \\
\leq \int_0^1 F^0(t, u(t); w(t)) dt,
\end{align*}
\]
for all \( w \in W^{2,p}(0, 1) \). Thus,
\[
\Phi^0(u; w) \leq \int_0^1 (-F^0(t, u(t); w(t)) dt + [\varphi(u), w],
\]
and from (22), we get
\[
\begin{align*}
\int_0^1 (-F^0(t, u(t); w(t)) dt - \int_0^1 F^0(t, u(t); w(t)) dt + f'(u; w) \\
\geq \left( |u''|^2 u'' \right) (1) - a(1)|u'(1)|^2 u'(1) w(1) - \left( |u''|^2 u'' \right) (0) - a(0)|u'(0)|^2 u'(0) w(0) \\
- |u''(1)|^2 u''(1) w'(1) + |u''(0)|^2 u''(0) w'(0),
\end{align*}
\]
(30)
for all \( w \in W^{2,p}(0, 1) \). Assume that \( x, y, z, q \in \mathbb{R} \) are arbitrary constants. Let \( w_n \in W^{2,p}(0, 1), n \in \mathbb{N} \), be defined by
\[
w_n := \begin{cases} 
\frac{x_0(t)}{n} + \frac{x}{n} \omega_1(t), & \text{if } t \in [0, \frac{1}{n}), \\
0, & \text{if } t \in [\frac{1}{n}, 1 - \frac{1}{n}], \\
\frac{z_0(n(1 - t))}{n} - \frac{z}{n} \omega_1(n(1 - t)), & \text{if } t \in (1 - \frac{1}{n}, 1].
\end{cases}
\]
where \( \omega_0(s) \) and \( \omega_1(s) \) are such that \( \omega_0(1) = \omega_1(1) = \omega_0'(1) = \omega_1'(1) = 0, \omega_0(0) = \omega_1(0) = 1 \) and \( \omega_0'(0) = \omega_1'(0) = 0 \), e.g., \( \omega_0(s) = (s - 1)^2(2s + 1) \) and \( \omega_1(s) = s(s - 1)^2 \). It is easy to check that \( w_n(0) = x, w_n(0) = y, w_n(1) = z \) and \( w_n(1) = q \).

Since hypothesis (H1) holds, there is \( \alpha_\rho \in L^1(0, 1) \) such that
\[
|F^0(t, u(t); \eta)| \leq \alpha_\rho(t)|\eta|, \quad \forall \eta \in \mathbb{R}, \text{ for a.a. } t \in (0, 1),
\]
where \( \rho > 0 \) depends on the supremum norm \( ||u||_\infty \) of \( u \). Thus, if in particular \( \eta = w_n(t) \), one obtains
\[
|F^0(t, u(t); w_n(t))| \leq \alpha_\rho(t) \max\left\{|x| + \frac{4|y|}{27n}, |z| + \frac{4|q|}{27n}\right\},
\]
(31)
for a.a. \( t \in (0, 1) \). Then, by Lebesgue’s dominated convergence theorem,
\[
\int_0^1 F^0(t, u(t); w_n(t)) dt \to 0, \quad \text{as } n \to \infty,
\]
(32)
in view of the fact that
\[
F^0(t, u(t); w_n(t)) \to F^0(t, u(t); 0) = 0, \quad \text{for a.a. } t \in (0, 1).
\]

Similarly, one has
\[
\int_0^1 (-F^0(t, u(t); w_n(t)) dt \to 0, \quad \text{as } n \to \infty.
\]
(33)
Finally, we take \( w = w_n \) in (30) and let \( n \to \infty \). Thus, by (23), (32) and (33), we derive
\[
\begin{align*}
f'((u(0), u(1), u'(0), u'(1))^T, (x, z, y, q)^T) & \geq \left( |u''|^2 u'' \right) (1) - a(1)|u'(1)|^2 u(1) z \\
- \left( |u''|^2 u'' \right) (0) - a(0)|u'(0)|^2 u'(0) x \\
- |u''(1)|^2 u''(1) q + |u''(0)|^2 u''(0) y.
\end{align*}
\]
Since \( x, y, z, \) and \( q \) are arbitrarily chosen, it follows that \( u \) satisfies (2). \( \square \)
Lemma 2.1. We have \( \lambda_1 > -\infty \), where \( \lambda_1 \) is the constant defined by (10). Moreover, if \( \lambda_1 > 0 \), then there exists a constant \( m > 0 \) such that
\[
\int_0^1 \left( |u''|^p + a|u'|^p + b|u|^p \right) dt \geq m\|u\|_{W^{2,p}(0,1)}^p, \quad \forall u \in \mathcal{D}.
\]

**Proof.** First, there exists a constant \( K \) such that
\[
\|u\|_{L^p}^p \leq K (\varepsilon \|u''\|_{L^p}^p + \varepsilon^{-1} \|u\|_{L^p}^p), \quad \forall \varepsilon \in (0, 1).
\]
Choose \( \varepsilon \) such that \( K\varepsilon \leq |a|^{-1} \) where \( |a| := \max_{t \in [0,1]} |a(t)| \). Let \( \lambda \) be such that \( b(t) + \lambda \geq |a|\varepsilon^{-1} \). Then,
\[
\|u\|_{L^p}^p \leq K (\varepsilon \|u''\|_{L^p}^p + \varepsilon^{-1} \|u\|_{L^p}^p) \leq |a|^{-1} \int_0^1 \left( |u''|^p + (b + \lambda)|u|^p \right) dt,
\]
i.e.,
\[
\int_0^1 \left( |u''|^p + a|u'|^p + b|u|^p \right) dt \geq \int_0^1 \left( |u''|^p + |a| |u'|^p + b|u|^p \right) dt \geq -\lambda \|u\|_{L^p}^p,
\]
so \( \lambda_1 \geq -\lambda > -\infty \).

Next, suppose that \( \lambda_1 > 0 \) holds. We will prove that there is some \( \mu > 0 \) such that
\[
\int_0^1 \left( |u''|^p + a|u'|^p + b|u|^p \right) dt \geq \frac{\mu}{\mu + 1} \left( \|u''\|_{L^p}^p + \|u'\|_{L^p}^p + \|u\|_{L^p}^p \right),
\]
for all \( u \in \mathcal{D} \). The inequality (35) is equivalent to
\[
\int_0^1 \left( |u''|^p + (a-1)\mu + a)|u'|^p + ((b-1)\mu + b)|u|^p \right) dt \geq 0.
\]
We will prove that there is \( \mu > 0 \) such that the following stronger inequality
\[
\int_0^1 \left( |u''|^p + (-|a|+1)\mu + a)|u'|^p + (-|b|+1)\mu + b)|u|^p \right) dt \geq 0
\]
holds for every \( u \in \mathcal{D} \). Let \( K \) be as above and \( \varepsilon, 0 < \varepsilon < 1 \) be such that \( K\varepsilon |a| < 1 \). We denote with \( \delta, 0 < \delta < 1 \) a number that will be chosen later. We have
\[
\int_0^1 \left( |u''|^p + (-|a|+1)\mu + a)|u'|^p + (-|b|+1)\mu + b)|u|^p \right) dt \\
\geq (1 - \delta) \int_0^1 \left( |u''|^p + a|u'|^p + b|u|^p \right) dt - (|b|+1)\mu \int_0^1 |u|^p dt \\
+ \delta \int_0^1 |u''|^p dt - (|a|+1)\mu \int_0^1 |u'|^p dt \\
\geq ((1 - \delta)\lambda_1 - (|b|+1)\mu) \int_0^1 |u|^p dt + \delta \int_0^1 |u''|^p dt - (|a|+1)\mu \int_0^1 |u'|^p dt
\]
for every \( u \in \mathcal{D} \). We look for \( \mu \) and \( \delta \) such that the inequalities
\[
\delta \geq (|a|+1)\mu \varepsilon, \\
A := (1 - \delta)\lambda_1 - (|b|+1)\mu \geq (|a|+1)\mu \varepsilon^{-1}
\]

(36)

(37)
are satisfied. The first inequality is equivalent to
\[ 1 - K \varepsilon |a|_\infty \delta \geq \mu \]  
(38)
and the second one is equivalent to
\[ \lambda_1 \geq (\lambda_1 + |b|_\infty + |a|_\infty K \varepsilon^{-1}) \delta + (1 + |b|_\infty + (|a|_\infty + 1) K \varepsilon^{-1}) \mu. \]  
(39)
Obviously there are \( \mu \) and \( \delta \) such that (38) and (39) hold as well as the inequalities (36) and (37). Therefore,
\[
\begin{aligned}
\int_0^1 \left( |u''|^p + (-(|a|_\infty + 1) \mu + a)|u'|^p + (-(|b|_\infty + 1) \mu + b)|u|^p \right) dt \\
\geq A \int_0^1 |u|^p dt + \delta \int_0^1 |u''|^p dt - (|a|_\infty \delta + (|a|_\infty + 1) \mu) \int_0^1 |u'|^p dt \\
\geq A \int_0^1 |u|^p dt + \delta \int_0^1 |u''|^p dt - (|a|_\infty \delta + (|a|_\infty + 1) \mu) \left( K \varepsilon \int_0^1 |u''|^p dt + K \varepsilon^{-1} \int_0^1 |u|^p dt \right) \geq 0.
\end{aligned}
\]
\[ \square \]

**Lemma 2.2.** Assume that (15) holds. Then \( \bar{x}_1 < +\infty \).

**Proof.** The inequality (15) implies that
\[ s^{-p} j(sz) \leq j(z) + \frac{k}{p} (1 - s^{-p}), \quad \forall z \in D(j), \quad \forall s \geq 1. \]
Then,
\[ \bar{x}_1 \leq \inf \{ \|u''\|^p_{L^p} + |a|_\infty \|u'\|^p_{L^p} + |b|_\infty \|u\|^p_{L^p} + p f(u) : u \in W^{2,p}(0, 1), \|u\|^p_{L^p} = 1, \] 
\[ (u(0), u(1), u'(0), u'(1))^T \in D(j) \} + k. \]  
\[ \square \]

**Proof of Theorem 1.1.** There are constants \( \sigma > 0 \) and \( \rho > 0 \) such that
\[ F(t, x) \leq \frac{\lambda_1 - \sigma}{p} |x|^p \quad \text{for a.a.} \ t \in (0, 1), \]
and for all \( x \) with \( |x| > \rho \), since inequality (12) holds. Hence,
\[ F(t, x) \leq \rho \alpha_\rho(t) + \left| \frac{\lambda_1 - \sigma}{p} \rho^p + \frac{\lambda_1 - \sigma}{p} |x|^p \right| \quad \text{for a.a.} \ t \in (0, 1), \ x \in \mathbb{R}, \]
which gives
\[ \Phi(u) - \varphi(u) = - \int_0^1 F(t, u) dt \geq -k - \frac{\lambda_1 - \sigma}{p} \int_0^1 |u|^p dt, \quad \forall u \in W^{2,p}(0, 1). \]  
(40)
On the other hand, \( f \) is proper, convex and l.s.c. Therefore, it is bounded from below by an affine functional, i.e.,
\[ \Phi(u) \geq -c_1 - c_2 \|u\|, \]  
(41)
with some constants \( c_1 > 0 \) and \( c_2 > 0 \). We use (40) and (41) to prove that functional \( f \) is coercive. Supposing on the contrary that \( \{u_n\} \subset W^{2,p}(0, 1) \) is a sequence such that \( \|u_n\|_{W^{2,p}(0, 1)} \to \infty \) and \( f(u_n) \leq C \) with a constant \( C > 0 \). Denote \( v_n := \frac{u_n}{\|u_n\|} \), where \( \| \cdot \| \) is the norm in \( W^{2,p}(0, 1) \). Then, \( u_n, v_n \in D(f) \). We have
\[
\begin{aligned}
C \geq \frac{1}{p} \int_0^1 \left( |u_n''|^p + a |u_n'|^p + b |u_n|^p \right) dt + \lambda_1 - \sigma \rho^p \int_0^1 |u_n|^p dt + F(u_n) - \varphi(u_n) \\
\geq \frac{1}{p} \int_0^1 \left( |u_n''|^p + a |u_n'|^p + b |u_n|^p \right) dt + \lambda_1 - \sigma \rho^p \int_0^1 |u_n|^p dt + F(u_n) - \varphi(u_n) - c_1 - c_2 \|u_n\|.
\end{aligned}
\]
which implies
\[
\frac{C + c_1}{\|u_n\|^p} + \frac{c_2}{\|u_n\|^p - 1} \geq \frac{1}{p} \int_0^1 \left( |u''_n|^p + a|u'_n|^p + b|u_n|^p \right) dt + \frac{\Phi(u_n) - \varphi(u_n)}{\|u_n\|^p}.
\]
\[
(42)
\]
Now, it follows
\[
\frac{C + c_1 + k}{\|u_n\|^p} + \frac{c_2}{\|u_n\|^p - 1} \geq \frac{1}{p} \int_0^1 \left( |u''_n|^p + a|u'_n|^p + (b - \lambda_1 + \sigma)|u_n|^p \right) dt
\]
\[
\geq \frac{\sigma}{p} \int |v_n|^p dt.
\]
\[
(43)
\]
Since $\|v_n\| = 1$, there exists a subsequence of $\{v_n\}$ (denoted again by $\{v_n\}$) and $v \in D(j)$, such that $v_n \to v$ in $W^{2,p}(0,1)$. Therefore, $v_n \to v$ strongly in $C^1([0,1])$. Taking into account estimate (43), we deduce that in fact $v_n \to v = 0$ in $C^1([0,1])$. Then,
\[
\|v''_n\|^p_{L^p} = \|v_n\|^p_{W^{2,p}} - \|v_n'\|^p_{L^p} - \|v_n\|^p_{L^p} \to 1, \quad \text{as} \quad n \to \infty.
\]
This implies that
\[
\frac{C + c_1 + k}{\|u_n\|^p} + \frac{c_2}{\|u_n\|^p - 1} \geq \frac{1}{p} \int_0^1 \left( |u''_n|^p + a|u'_n|^p + (b - \lambda_1 + \sigma)|u_n|^p \right) dt \to \frac{1}{p}
\]
a contradiction. The coercivity of functional $I$ is verified.

Next, the functional $\Phi$ is sequentially weakly continuous due to the compactness of the imbedding $W^{2,p}(0,1) \subset C^1([0,1])$. Hence, by the convexity of $\psi$, functional $I$ is sequentially weakly lower semi-continuous and its coercivity implies that it is bounded from below and attains its infimum. Then $I$ has a critical point, which by Theorem 2.2 is a solution of problem (1)–(2). □

**Lemma 2.3.** Assume $(H_1)$ holds, $\lambda_1 > 0$, and either $(G_0)$ or $(G_p)$ holds. If, in addition, $D(j)$ is closed, then functional $I$ satisfies the Palais–Smale condition.

**Proof.** Let $\{u_n\}$ be an arbitrary Palais–Smale sequence. Set $v = (1 + s)u_n$ in the inequality (18), where $s > 0$. Taking the limit as $s \to 0^+$, we obtain
\[
\Phi^0(u_n; u) + \psi'(u_n; u) \geq -\varepsilon_n \|u_n\|.
\]
This inequality reads
\[
\Phi^0(u_n; u) \leq \langle \psi'(u_n), u_n \rangle + \frac{1}{p} \int_0^1 \left( |u''_n|^p + a|u'_n|^p + b|u_n|^p \right) dt + f'(u_n; u) \geq -\varepsilon_n \|u_n\|.
\]
\[
(44)
\]
Next, there exists a constant $C$ such that
\[
C \geq I(u_n) = \frac{1}{p} \int_0^1 \left( |u''_n|^p + a|u'_n|^p + b|u_n|^p \right) dt + f(u_n) + \Phi(u_n) - \varphi(u_n).
\]
\[
(45)
\]
We use (44) and (45) to prove that in fact the sequence $\{u_n\}$ is bounded. We will examine separately the cases when $(G_p)$ and $(G_0)$ hold.

**Case 1.** Let $(G_0)$ hold for some $\theta > p$. First, we verify that
\[
\theta \left( \Phi(u) - \varphi(u) \right) \geq \Phi^0(u; u) - \langle \psi'(u), u \rangle - m_1, \quad \forall u \in W^{2,p}(0,1),
\]
\[
(46)
\]
and
\[
\theta f(u) \geq f'(u; u) - m_2, \quad \forall u \in W^{2,p}(0,1),
\]
\[
(47)
\]
for some positive constants $m_1$ and $m_2$. The inequality (47) follows from the definition of the functional $f$ and condition (13). Now, let $I \in \delta \Phi(u)$. Then, there exists $u_i \in L^1$ (see Proposition 2.1), such that $u_i(t) \in \delta F(t, u(t))$ for a.a. $t \in (0,1)$, and
\[ \langle l, v \rangle = \int_0^1 \left( -u(t)v(t) + (a(t) - 1)|u'(t)|^{p-2}u'(t)v(t) + (b(t) - 1)|u(t)|^{p-2}u(t)v(t) \right) dt, \quad \forall v \in W^{2,p}(0, 1). \]

Next, hypothesis \((H_1)\) says that given \(M > 0\) there exists an \(\alpha_M(t) \in L^1\) such that for each \(x \in \mathbb{R}\), with \(|x| \leq M\), the inequalities
\[ |\xi| \leq \alpha_M(t), \quad \forall \xi \in \bar{\partial}F(t, x), \]
and
\[ |F(t, x)| \leq M\alpha_M(t), \]
are satisfied. Hence,
\[
\int_0^1 u(t)u_1(t) dt = \int_{|u(t)| > M} u(t)u_1(t) dt + \int_{|u(t)| \leq M} u(t)u_1(t) dt \\
\geq \int_{|u(t)| > M} \theta F(t, u(t)) dt - M \int_0^1 \alpha_M(t) dt \\
= \theta \left( \int_0^1 F(t, u(t)) dt - \int_{|u(t)| \leq M} F(t, u(t)) dt \right) - M \int_0^1 \alpha_M(t) dt \\
\geq \theta \int_0^1 F(t, u(t)) dt - M(1 + \theta) \int_0^1 \alpha_M(t) dt.
\]

Obviously,
\[
\int_0^1 F(t, u(t)) dt = -\Phi(u) + \varphi(u) \quad \text{and} \quad \int_0^1 u(t)u_1(t) dt = -\langle l, u \rangle + \langle \varphi'(u), u \rangle
\]
and we get
\[ \theta \left( \Phi(u) - \varphi(u) \right) \geq \langle l, u \rangle - \langle \varphi'(u), u \rangle - m_1, \quad \forall l \in \bar{\partial} \Phi(u), \]
where \(m_1 = M(1 + \theta) \int_0^1 \alpha_M(t) dt > 0\). Finally, it follows that
\[ \theta \left( \Phi(u) - \varphi(u) \right) \geq \max \{ \langle l, v \rangle : l \in \bar{\partial} \Phi(u) \} - \langle \varphi'(u), u \rangle - m_1 \]
\[ = \Phi^0(u; u) - \langle \varphi'(u), u \rangle - m_1, \]
which yields (46).

Now, setting \(u = u_n\) in (46) and (47) and multiplying (45) by \(\theta\), one can derive from (44)–(47) that
\[ \theta C + m_1 + m_2 \geq \left( \frac{\theta}{p} - 1 \right) \int_0^1 \left( |u''|^{p} + a|u'|^{p} + b|u_n|^{p} \right) dt - \epsilon_n \|u_n\|. \]

Finally, by the hypothesis \(\lambda_1 > 0\) and by Lemma 2.1, there exists a constant \(m_3 > 0\) such that
\[ \theta C + m_1 + m_2 \geq m_3 \|u_n\|^p - \epsilon_n \|u_n\|, \]
which implies that \(\{u_n\}\) is bounded.

**Case 2.** Let \((G_p)\) hold. First, using hypothesis \((16)\) we derive a similar to (46) inequality. More precisely, we show that there exists a constant \(k_1 > 0\), and, given \(\rho > 0\) there exists a constant \(m_1 = m_1(\rho) > 0\), such that for each \(u \in W^{2,p}(0, 1)\), with \(\|u\| \geq \rho\), the following
\[ \left( p + \frac{k_1}{\|u\|^{p-1}} \right) \left( \Phi(u) - \varphi(u) \right) \geq \Phi^0(u; u) - \langle \varphi'(u), u \rangle - m_1 \]
holds. Let \(l \in \bar{\partial} \Phi(u)\) and \(u_1 \in L^1\) be defined as in Case 1. Then,
where the positive constant $d$ is such that $|u(t)| \leq d\|u\|_{W^{2,p}(0,1)}$. Similarly as in Case 1, we obtain that (48) holds with $k_1 = c/d$, and
\[
m_1 = M \left( p + 1 + \frac{c}{d\rho^{-1}} \right) \int_0^1 \alpha_M(t) \, dt.
\]

Next, inequalities (15) and (41) imply
\[
\left( p + \frac{k_1}{\|u\|^{p-1}} \right) f'(u) \geq f'(u; u) - k + \frac{k_1}{\|u\|^{p-1}} f(u)
\]
\[
\geq f'(u; u) - k - k_1 \|u\|^{p-1} + c_1, \quad \forall u \in W^{2,p}(0,1), \|u\| \geq 1.
\]

(49)

We are ready to prove that the sequence $\{u_n\}$ is bounded. Suppose on the contrary that $|u_n| \to \infty$. We may assume that $\|u_n\| \geq 1$ for all $n$. Then, applying (48) with $\rho = 1$ and (49) to (45) and (44), we get
\[
pC + \tilde{C} + m_1 \geq \left( \frac{k_1 m}{p} - \frac{k_1 c_2}{\|u_n\|^{p-1}} - \varepsilon_n \right) \|u_n\|.
\]

for some constants $\tilde{C} > 0$ and $m_1 > 0$. Now, by $\lambda_1 > 0$ and Lemma 2.1, inequality (50) implies
\[
pC + \tilde{C} + m_1 \geq \left( \frac{k_1 m}{p} - \frac{k_1 c_2}{\|u_n\|^{p-1}} - \varepsilon_n \right) \|u_n\|,
\]
a contradiction since $\varepsilon_n \to 0$ and $\|u_n\| \to \infty$. Therefore $\{u_n\}$ is bounded.

Next, let $\{u_n\}$ again be a Palais-Smale sequence. Since $\{u_n\}$ is bounded under each of the hypotheses $(G_\rho)$ and $(G_\mu)$, there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, and $u \in W^{2,p}(0,1)$, such that $u_n \rightharpoonup u$ in $W^{2,p}(0,1)$. Thus, $u_n \to u$ (for a subsequence) strongly in $C^1([0,1])$ and, since $D(j)$ is convex and closed, then $u \in D(j)$. From (18) we derive that
\[
\Phi^0(u_n; u - u_n) + f'(u_n; u - u_n) + \varepsilon_n \|u - u_n\| \geq - \int_0^1 \left( |u_n'|^{p-2} u_n' u'' + |u_n'|^{p-2} u_n u' + |u_n|^{p-2} u_n u \right) \, dt + \|u_n\|^p
\]
\[
\geq \|u_n\|^{p-1} \left( \|u_n\| - \|u\| \right),
\]
yielding that
\[
\Phi^0(u_n; u - u_n) + f'(u_n; u - u_n) + \varepsilon_n \|u - u_n\| - \|u\|^{p-1} \left( \|u_n\| - \|u\| \right)
\]
\[
\geq \left( \|u_n\|^{p-1} - \|u\|^{p-1} \right) \left( \|u_n\| - \|u\| \right).
\]

(51)
Functional $\Phi$ has the trivial extension $\tilde{\Phi}$ on the space $C^1([0, 1])$ defined by (19). Moreover, $\tilde{\Phi}$ is locally Lipschitz functional and obviously

$$\Phi^0(v; w) = \tilde{\Phi}^0(v; w), \quad \forall v, w \in W^{2, p}(0, 1).$$

The upper semi-continuity of $\tilde{\Phi}^0(\cdot; \cdot)$ yields

$$\limsup_{n \to \infty} \Phi^0(u_n; u - u_n) \leq \tilde{\Phi}^0(u; 0) = 0.$$  \hspace{1cm} (52)

On the other hand,

$$\limsup_{n \to \infty} f'(u_n; u - u_n) \leq \limsup_{n \to \infty} (f(u) - f(u_n)) = f(u) - \liminf_{n \to \infty} f(u_n) \leq 0.$$  \hspace{1cm} (53)

Hence, taking into account (51)–(53), we obtain

$$0 \geq \limsup_n (\|u_n\|^p - 1 - p^{-1}\|u\|^p)(\|u_n\| - \|u\|)$$

implying that $\|u_n\| \to \|u\|$. Since $(W^{2, p}(0, 1), \|\cdot\|)$ is uniformly convex, $u_n \to u$ strongly in $W^{2, p}(0, 1)$. Thus, $I$ satisfies the Palais–Smale condition, as claimed. \hfill \Box

**Proof of Theorem 1.2.** We will show that functional $I$ satisfies the hypotheses of the mountain pass Theorem 2.1. First of all, according to Lemma 2.3, it satisfies the Palais–Smale condition. Since $(0, 0, 0, 0)^T \in \partial j((0, 0, 0, 0)^T)$, we have

$$I(u) \geq j(0) = j((0, 0, 0, 0)^T).$$

We assume without any loss of generality that $j((0, 0, 0, 0)^T) = 0$ and so in particular $I(0) = 0$. Next, we will prove that there exist constants $\rho > 0$ and $\alpha > 0$ such that $I(u) \geq \alpha$ for all $u \in W^{2, p}(0, 1)$ such that $\|u\| = \rho$. Here $\|\cdot\|$ denotes as usual the norm of $W^{2, p}(0, 1)$. More precisely, there exist constants $\sigma > 0$ and $\delta > 0$, such that

$$F(t, x) \leq \frac{\lambda_1 - \sigma}{p} |x|^p, \quad \forall |x| \leq \delta.$$  \hspace{1cm} (54)

Next, there exists a constant $d > 0$ such that $|u(t)| \leq d\|u\|$ for all $u \in W^{2, p}(0, 1)$. Hence, if $\rho = \|u\| < d^{-1}\delta$, then inequality (54) can be applied with $x$ replaced by $u(t)$. We have

$$I(u) = \frac{1}{p} \int_0^1 (|u''|^p + a|u'|^p + b|u|^p) dt - \int_0^1 F(t, u) dt + f(u)$$

$$\geq \frac{1}{p} \int_0^1 (|u''|^p + a|u'|^p + b|u|^p) dt - \frac{\lambda_1 - \sigma}{p} \int_0^1 |u|^p dt \geq m\|u\|^p,$$

for some constant $m > 0$. Here, we have used the inequality

$$\int_0^1 (|u''|^p + a|u'|^p + (b - \lambda_1 + \sigma)|u|^p) dt \geq \sigma \int_0^1 |u|^p dt,$$

which is an immediate consequence of Lemma 2.1 for $b$ replaced with $b - \lambda_1 + \sigma$.

Finally, we need to find a function $e \in W^{2, p}(0, 1)$ such that

$$I(e) \leq 0 \quad \text{and} \quad \|e\| > \rho.$$  \hspace{1cm} (55)

In what follows, we will examine separately the two alternative cases of the theorem.

**Case 1.** Let $(G_\alpha)$ hold. Assume that $|x| > M$, where $M$ is the constant which appear in the hypotheses of the theorem. The mapping $s \mapsto s^{-\alpha}F(t, sx)$ is locally Lipschitz for $a.a. t \in (0, 1)$, so we have for each $s > 0$

$$s^{-\alpha}F(t, sx) \subset \overline{\partial_s(s^{-\alpha}F(t, sx))} = s^{-\alpha-1}(-\partial F(t, sx) + sx\tilde{\alpha}F(t, sx)).$$

Given $1 \leq r < s$, by Lebourg's mean value theorem and assumption (14), there exist $\tau \in (r, s)$ and $\xi \in \overline{\partial_s(s^{-\alpha}F(t, sx))}|_{s=\tau}$, $\xi \geq 0$, such that

$$s^{-\alpha}F(t, sx) - r^{-\alpha}F(t, rx) = \xi(s - r) \geq 0.$$
i.e.,
\[ F(t, sx) \geq s^p F(t, x), \quad \text{for a.a.} \ t \in [0, 1], \ \forall |x| > M, \ s \geq 1. \]

Now, let \( h \in C_0^\infty(0, 1) \) be such that \( |h| > M \) on a set with positive measure. Then,
\[
\frac{1}{p} \int_0^1 F(t, sh) \, dt = \int_{\{|h| > M\}} F(t, sh) \, dt + \int_{\{|h| \leq M\}} F(t, sh) \, dt
\]
\[
\geq \int_{\{|h| > M\}} F(t, sh) \, dt - M \int_0^1 \alpha_M(t) \, dt
\]
\[
\geq s^p \int_{\{|h| > M\}} F(t, h) \, dt - M \int_0^1 \alpha_M(t) \, dt,
\]
for all \( s \geq 1 \). We have \( J(sh) = 0 \) for each \( s \), thus
\[
I(sh) = \frac{s^p}{p} \int_0^1 \left( |h''|^p + a|h'|^p + b|h|^p \right) \, dt - \int_0^1 F(t, sh) \, dt
\]
\[
\leq \frac{s^p}{p} \int_0^1 \left( |h''|^p + a|h'|^p + b|h|^p \right) \, dt - s^p \int_{\{|h| > M\}} F(t, h) \, dt + M \int_0^1 \alpha_M(t) \, dt
\]
for all \( s \geq 1 \). The latter inequality reads
\[
I(sh) \leq s^p k_1 - s^p k_2 + k_3 \to -\infty, \quad \text{as} \ s \to \infty,
\]
with constants \( k_1, k_2, k_3 > 0 \). Finally, take \( s_0 \) sufficiently large such that \( I(s_0 h) \leq 0 \) and \( \|s_0 h\| > \rho \). Then \( e := s_0 h \) satisfies conditions (55).

**Case 2.** Let \((G_p)\) and \((I_{\infty})\) hold. Let \( u_n \in \mathcal{D} \) and \( s_n > 0 \) be such that \( \|u_n\|_{L^p} = 1, \ s_n \to \infty \).
\[
\int_0^1 \left( |u''|^p + a|u'|^p + b|u|^p \right) \, dt + \frac{p J(s_n u_n)}{s_n^p} \to \lambda_1.
\]
Condition \((I_{\infty})\) implies that there exist constants \( C > 0 \) and \( \sigma > 0 \) such that
\[
F(t, x) \geq \frac{\lambda_1 + \sigma}{p} |x|^p, \quad \forall |x| > C, \ \text{a.a.} \ t \in (0, 1).
\]
We have
\[
\frac{1}{p} \int_0^1 F(t, s_n u_n(t)) \, dt = \int_{\{|s_n u_n| > C\}} F(t, s_n u_n) \, dt + \int_{\{|s_n u_n| \leq C\}} F(t, s_n u_n) \, dt
\]
\[
\geq \frac{s_n^{p \lambda_1 + \sigma}}{p} \int_{\{|s_n u_n| > C\}} |u_n|^p \, dt - C \int_0^1 \alpha_c(t) \, dt
\]
\[
= \frac{s_n^{p \lambda_1 + \sigma}}{p} \left( \int_0^1 |u_n|^p \, dt - \int_{\{|s_n u_n| \leq C\}} |u_n|^p \, dt \right) - C \int_0^1 \alpha_c(t) \, dt
\]
\[
\geq \frac{s_n^{p \lambda_1 + \sigma}}{p} \left( \frac{\lambda_1 + \sigma}{p} |C|^p - C \int_0^1 \alpha_c(t) \, dt \right).
\]
Hence

$$I(s_n u_n) \leq \frac{s_n^p}{p} \int_0^1 \left( \left| u_n'' \right|^p + a \left| u_n' \right|^p + b |u_n|^p \right) \, dt + J(s_n u_n) - \frac{s_n^p}{p} \left( \lambda_1 + \sigma \right) - C^p - C \int_0^1 \alpha_C(t) \, dt.$$ 

Therefore,

$$\frac{I(s_n u_n)}{s_n^p} \leq \frac{1}{p} \int_0^1 \left( \left| u_n'' \right|^p + a \left| u_n' \right|^p + b |u_n|^p \right) \, dt + \frac{J(s_n u_n)}{s_n^p} - \frac{\left( \lambda_1 + \sigma \right)}{p} - \frac{C_1}{s_n^p},$$

which converges to $-\sigma/p$ as $n \to \infty$. Finally, let $n$ be such that $I(s_n u_n) < 0$ and $\|s_n u_n\| > \rho$. Obviously, $e := s_n u_n$ satisfies (55). □

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References