The Quantified Argument Calculus

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ABSTRACT. I develop a formal logic in which quantified arguments occur in argument positions of predicates. This logic also incorporates negative predication, anaphora and converse relation terms, namely, additional syntactic features of natural language. In these and additional respects it represents the logic of natural language more adequately than does any version of Frege's Predicate Calculus. I first introduce the system's main ideas and familiarise it by means of translations of natural language sentences. I then develop a formal system built on these principles, the Quantified Argument Calculus or Quarc. I provide a truth-value assignment semantics and a proof system for the Quarc. I next demonstrate the system's power by a variety of proofs; I prove its soundness; and I comment on its completeness. I then extend the system to modal logic, again providing a proof system and a truth-value assignment semantics. I proceed to show how the Quarc versions of the Barcan formulas, of their converses and of necessary existence come out straightforwardly invalid, which I argue is an advantage of the modal Quarc over modal Predicate Logic as a system intended to capture the logic of natural language.

I. Introduction

In a series of publications over the last decade (Ben-Yami 2004, 2009a, 2009b, 2012; Lanzet and Ben-Yami 2004) I have developed a system of logic which is closer in some respects to Aristotle's logic than it is to Frege's calculus, and which incorporates ideas found in the works of Strawson (1952) and Geach (1962). Like systems of plural logic developed following Boolos's work (1984), my system also allows plural terms to occur in argument position. However, unlike these plural logics, its treatment of quantification is quite unlike that of the Predicate Calculus: quantifiers do not combine with variables and operate on sentential functions. Rather, in this system, quantifiers combine with one-place predicates and the expressions formed in this way occur in argument positions. This system can therefore be called the Logic of Quantified Arguments. I have argued in the works mentioned above that it offers a better analysis of the logic of natural language than does the Predicate Calculus, its generalised quantifier version and Boolosian plural logic variants included.

Although a formal system that incorporates the principles of the Logic of Quantified Arguments was partly introduced in (Ben-Yami 2004, 2009a), these works do not contain a fully developed formal system. Such a system or calculus is presented in (Lanzet and Ben-Yami 2004), but the Logic of Quantified Arguments allows for a calculus less complex and more elegant in several respects than the one introduced there. Moreover, on the one hand (Lanzet and Ben-Yami 2004) contains some discussions belonging to the philosophy of language which are unnecessary for the presentation of the formal system; and, on the other hand, much of the material there is squarely within mathematical logic and is unnecessary in a presentation of the calculus intended for a wider audience, an audience which may include philosophers and linguists who use formal logic in their work.

A main purpose of this paper is to introduce a calculus for the Logic of Quantified Arguments, the *Quantified Argument Calculus* or *Quarc*. As the Logic of

Quantified Arguments has been applied to a variety of issues that I shall not try to incorporate in the calculus in this paper—for instance, plural arguments, proportionality quantifiers ('most', 'many') and non-distributive predication[1]—and as logic need not always use an artificial formal system, I find it helpful to distinguish here the logic from the calculus. In addition, as a rule I shall not try to explain, as part of the presentation of the system, why I think the Logic of Quantified Arguments with its calculus offer a better analysis of the logic of natural language than does the Predicate Calculus on any of its available versions. Some such advantages will become evident in the course of the paper, but I shall not elaborate on them here. This has been done in the works mentioned above. In this paper, I keep discussions pertaining to philosophy of language to a minimum.

However, after introducing and developing the Quarc in the next two sections, I extend it in the fourth section to modal logic. As I did not discuss the application of the Logic of Quantified Arguments to modal logic in any earlier publication, I emphasise in that section some of the advantages of the system over the Predicate Calculus in capturing some features of the modal idiom of natural language.

Since not all readers will be familiar with the basic ideas of the system presented here, which in some respects departs significantly from the Predicate Calculus, the second section contains an introduction into the system, assisted by translations from natural language into the Quarc. I then systematically present the Quarc in the third section.

The formal system presented in this paper is primarily intended as paralleling the classical version of the Predicate Calculus. For this reason, I do not discuss below some important topics that demonstrate the power of the Logic of Quantified Arguments. For instance, the only quantifiers introduced into the calculus below are the universal and existential (or particular) ones, for other quantifiers are not included in standard versions of Fregean (as well as Aristotelian) logic. This restriction to the mentioned quantifiers is despite the fact that the Logic of Quantified Arguments can incorporate many more quantifiers, and the fact that the quantifiers it can incorporate match those actually realised in natural language better than what can be achieved by either the standard version of the Predicate Calculus or its generalised quantifier version (see Ben-Yami 2009b and the ensuing discussion between Westerståhl and Ben-Yami in issue 217 of *Logique et Analyse*). For a similar reason I do not apply the formal system developed below to non-distributive predication, notwithstanding the fact that this kind of predication was among the motivations for developing plural logics (see Ben-Yami 2004, McKay 2006 and Oliver and Smiley 2013). Standard versions of the Predicate Calculus necessarily incorporate only distributive predication, for they do not allow plural terms into argument position. For a somewhat different reason I do not introduce in this paper identity into the Quarc. The Quarc need not depart from the Predicate Calculus in its treatment of identity, and there is no point in rehearsing here familiar formal approaches that do not contribute to the new one I wish to develop.

Other attempts to develop logic calculi closer in various respects to the logic of natural language than is the Predicate Calculus are found in the literature. I have discussed the work of Fred Sommers, contained mainly in his book *The Logic of Natural Language* (1982), and compared it to mine in (Ben-Yami 2004, 2006). Here I shall relate my work to more recent ones, that of Lawrence Moss and his collaborators (Moss 2010;

[1] Predication is distributive just in case it applies to several items if and only if it also applies to each of them; otherwise it is non-distributive. In 'The children slept' predication is distributive, for if the children are Alice and Bob then the sentence is true if and only if 'Alice slept' and 'Bob slept' are also true; but in 'The children formed a circle' the predication is non-distributive, for if Bob is one of the children 'Bob formed a circle' need not be true.

Pratt-Hartmann and Moss 2009) and that of Nissim Francez (unpublished). Unlike me, Moss does not consider quantifiers as combining with one-place predicates and forming a quantified argument, but he takes them to be binary operators. The system that results in this way is significantly different from mine, and, I think, more remote from natural language. There are other points of similarity and of difference, and I shall mention those I take to be more significant in footnotes below and explain the reasons for my differing decisions. Francez, unlike Moss, does conceive of the quantifier the way I do, as combining with a one-place predicate or noun and forming the argument of a verb or predicate (he considers quantifiers *subnectors*); in this respect, our fundamental departure from the Predicate Logic is the same. However, the syntax he then develops, the additional features he incorporates into his system and his decisions along the way are quite unlike mine, and the resulting system is eventually quite different. Here too I shall note important points of similarity and difference along the paper.[2]

With these preliminaries in place, I proceed to introduce the system.

II. Introduction of the Logic of Quantified Arguments and Its Calculus

I shall translate the sentence

1. John is tall

into the Quarc not by T(j), as is done in the Predicate Calculus, but by writing the name or constant to the left of the predicate, (j)T. Or, omitting parentheses,

2. jT

I write the name to the left of the predicate for two reasons. First, this would help to prevent falling into old Predicate Calculus habits of thought. Secondly, it is closer to natural language, and similarity to natural language in word order would help to make the formulas introduced below, which make use of quantified arguments and copulative structure, easier to understand.

Many-place predicates are translated along the same lines:

3. John loves Mary
4. (j,m)L

It is possible to drop parentheses in this case too, as well as the comma, and write without ambiguity jmL; but to make formulas more easily surveyable I shall not do that in this paper.[3]

A main innovation of the Quarc is that it contains quantified arguments. In

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[2] A different kind of project is the development of a controlled fragment of some natural language, namely a precisely defined subset of, say, English; such a fragment can then be translated automatically and unambiguously into an appropriate logic calculus. See, for instance, (Fuchs *et al.* 2008) on ACE and (Schwitter 2010) on PENG and CPL; an introduction to the subject with a survey of existing controlled languages and many useful links can be found at https://sites.google.com/site/controllednaturallanguage/. A controlled fragment, however, is not committed to any specific logic, and the same controlled fragment might be translatable into several different logic systems. Because of this different nature of the controlled language project, it will not be considered below.

[3] Already in this my formulation is different from Moss's: Moss adopts a subject–predicate structure and constructs a relation with its object as a complex predicate, while I adopt the many-place predicate interpretation of relations, as is done by the Predicate Calculus.

5. All men are tall

'All men' is analysed as the argument of the predicate 'tall', the way 'John' is in 'John is tall'. 'All men' is translated by '∀M', and the sentence as a whole is translated by

6. (∀M)T

We can omit the parentheses and write ∀MT, as I shall usually do below. It can also be seen that the grammar or syntax of this calculus is closer to that of natural language than is that of the Predicate Calculus.[4]

Similarly,

7. Some men are tall

is translated by

8. ∃MT

The word 'some' stands for the quantifier called *particular* in Aristotelian logic, a quantifier this logic did not identify with the existential construction 'there is' (see Ben-Yami 2004, § 6.5 for the distinction). Only with Frege did we come to identify the two (1879, § 12), and this despite their marked differences across natural languages. Since I think they should indeed be distinguished, and since of the two only the particular quantifier joins a general term to form a noun phrase that can occur in the argument position, it could be helpful to introduce a special symbol for the particular quantifier. A vertical mirror image of the letter P could serve this function. But as this symbol is not available on common character charts, and as the Predicate Calculus does use the symbol ∃ to translate particular quantification, I shall continue to use ∃ in this paper.

The formalisation of quantified sentences with many-place predicates is a straightforward extension of that of one-place predicates. For example,

9. Some men love all women

is translated by

10. (∃M,∀W)L

Again we can without ambiguity omit parentheses and write ∃M∀WL. I shall do that occasionally in this paper, in places in which it will not make the structure of the sentence unclear.

Natural language distinguishes affirmative from negative modes of predication, the latter usually indicated by means of a negative copula, as exemplified in the contrast between 'John is tall' and 'John isn't tall'. Unlike the Predicate Calculus, Aristotelian logic

[4] I mentioned above that Moss constructs quantifiers as binary operators and does not consider the quantified phrase as an argument of predicates. For this reason, singular terms (the constants of (Moss 2010, § 2.1)), which *are* for him arguments of predicates ('unary atoms', ibid.), don't have the same syntactic position in his system as quantified noun phrases (which are not a syntactic category on his approach). In having both singular and quantified arguments, my system is closer to natural language syntax than his.

followed natural language in this, and so shall the Quarc.[5] I shall also distinguish between negative predication and sentence negation. The respective sentences are translated as follows:

11. John isn't tall
12. j¬T
13. It's not the case that John is tall
14. ¬jT

To make the syntax clearer, sentence (12) can be written with the use of parentheses as (j)¬T and sentence (14) as ¬((j)T). The logical relations between predication negation and sentence negation will be characterised below by the derivation rules and the rules for truth-value assignments for negation. Although when all arguments are singular terms the two forms are equivalent, this is not the case when quantified arguments are used. The following natural language sentences are not equivalent, and neither are their translations:

15. Some men aren't tall
16. ∃M¬T
17. It's not the case the some men are tall
18. ¬∃MT

With these examples, we can introduce a notion of quantifier scope into the calculus. Since quantifiers are not sentential operators, this should be done differently than in the Predicate Calculus. To emphasise the difference, I shall not talk of scope but say that a quantified argument *governs* a sentence. Governance is defined below, while at this stage I clarify it by examples. In sentences (6), (8) and (16) the quantified argument ∃M governs the sentence, but not so in sentence (18), where it occurs in a sentence on which negation operates. In sentence (10), ∃M and not ∀W governs the sentence: to govern a sentence, a quantified argument must be the leftmost quantified argument in that sentence.[6]

In English, when we wish to say of each and every particular of a specific group that it is *not* of a certain sort, we use affirmative predication with the negative quantifier 'no':

19. No man is tall

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[5] Moss introduces instead of a negative mode of predication negative predicates, 'non-pineapple' for instance. That he feels this to be a departure from natural language, whose syntax he tries to follow (Pratt-Hartmann and Moss 2008: 648, 1st line) is clear from the question-and-answer Introduction to (Moss 2010), in which the questioning novice sarcastically remarks on this noun: '"non-pineapple"?! I thought this was supposed to be natural language.' Moss's mouthpiece responds that this noun should be taken as an abbreviation for 'piece of fruit which is not a pineapple'.—Yet this cannot be so, for in Moss's formal system negative nouns don't play the role of an abbreviation of anything else. Moreover, it would make Moss's claim there that everything is either a pineapple or a non-pineapple false, for something that is not a piece of fruit would be neither.

Apart from this departure from natural language, introducing modes of predication turns out to have great logical power when modes additional to affirmation and negation are considered, such as modal ones. This will become clear in the last section.

[6] This is the common case in natural language as well, although natural language admits exceptions to this rule. In this respect my treatment of governance in this paper contains a measure of regimentation.

A negative quantifier can be introduced into the calculus developed in this paper, but I shall not do that, for several reasons. First, I am primarily trying to develop a calculus that parallels the Predicate Calculus, and the latter does not use a negative quantifier. Rather, the Predicate Calculus paraphrases (19) by means of the universal quantifier:

20. $\forall x(Mx \rightarrow \neg Tx)$

Secondly, unlike English, some languages use in their translation of (19) negative predication with universal quantification, expressing universality with a quantifier-word specific for universal negative predication. This is the case, for instance, in French and Hebrew:

21. Aucun homme n'est grand
22. אף אדם אינו גבוה

The Quarc need not be more faithful to the English idiom than it is to that of other languages. Thirdly, limiting ourselves to two quantifiers would of course simplify the calculus. I shall therefore translate sentence (19) by a construction paralleling the paraphrase of (20) and closer to that of (21) and (22):[7]

23. $\forall M \neg T$

Next,

24. Not all man are tall

is translated by

25. $\neg \forall MT$

in which the negation is sentence negation. $\forall M$ governs sentence (23), but not (25).[8]

So far I have not used in the calculus anything which resembles variables. However, I shall next introduce into the Quarc *anaphors*, whose role, following the one anaphors have in natural language, *partly* overlaps with that of variables in the Predicate Calculus.[9] Unlike variables, anaphors are used in non-quantified sentences as well, for instance:

26. John loves *himself*
27. If John meets Mary, *he* greets *her*

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[7] Moss also paraphrases 'no' by means of the universal quantifier (Pratt-Hartmann and Moss 2008: 649, 2nd line), as does Francez.

[8] Some linguistic treatments view 'not all' and 'not every' as quantifiers on their own. But this is implausible, for if they were such we could say, for instance, 'John kissed not every girl', yet this sentence is ungrammatical. We say, instead, 'John didn't kiss every girl' or 'John kissed only some girls'. My analysis of 'not' in (24) as indicating sentential negation is therefore more plausible; and in any case, it is at least acceptable.

[9] For a comparison and contrast of variables and anaphors, see (Ben-Yami 2004: Chap. 8). Francez does not incorporate anaphors in his system, but uses standard variables instead; in this respect my system is closer to the syntax of natural language than is his and also captures more features of natural language syntax.

The anaphor in these cases has its reference determined as that of the expression it is anaphoric on, its source, in the sense that it could be substituted by its source (it is what Geach called 'pronoun of laziness' (1962: § 76)). Natural language indicates the anaphor's source by gender agreement, by using definite descriptions, by phrases like 'the former' and 'the latter', and so on. I shall use small Greek letters for anaphors, and the anaphor's source will be indicated by writing the anaphor as a subscript next to it. The translations of the last two sentences are therefore:

28. $(j_\alpha,\alpha)L$
29. $(j_\alpha,m_\beta)M \rightarrow (\alpha,\beta)G$

The same convention will be followed when the anaphor's source is a quantified noun phrase:

30. Every man loves himself
31. $(\forall M_\alpha,\alpha)L$

We next consider the analysis and translation of sentences containing donkey anaphora, for instance:

32. If John buys a donkey, he vaccinates it

These sentences used to be translated into the Predicate Calculus as if the indefinite noun phrase in the antecedent has universal force, namely,

33. $\forall x((\text{Donkey}(x) \ \& \ \text{Buy}(j,x)) \rightarrow \text{Vaccinate}(j,x))$

Evans, however, has argued that this interpretation of the indefinite article is unjustified (1977), and he and others have tried to provide different analyses of donkey sentences. By contrast, in my book I have tried to justify the earlier 'universal force' analysis and show why the indefinite article has to have that force in such conditional sentences (2004: § 8.4). This is the analysis I shall adopt in this paper. This is obviously not the place to justify it; but as even those who object to it agree that it gives the correct truth conditions of sentences like (32) (see Neale 2006: 364), anyone who is sceptical of the claim that the indefinite has universal force in such sentences may consider my translation as at least supplying a paraphrase with the same truth conditions as the original.

Sentence (32) will thus be translated as

34. $(j_\alpha,\forall D_\beta)B \rightarrow (\alpha,\beta)V$

$\forall D$, having an anaphor in the consequent, governs the whole conditional. Another example:

35. If a bishop meets a bishop, he greets him
36. $(\forall B_\alpha,\forall B_\beta)M \rightarrow (\alpha,\beta)G$

The first occurrence of $\forall B$ governs the conditional. A last example with anaphors:

37. If a Butcher bUys a Donkey, the donkey is dOomed.

38. $(\forall B,\forall D_{\alpha})U \rightarrow \alpha O$

We need to generalise the concept of governance to sentences like (38). In it, $\forall B$ governs the *whole* sentence: it is the leftmost quantified argument in that sentence, and (38) does not contain any sub-sentence S' which contains $\forall B$ as well as all anaphors of all quantified arguments occurring in it (in S'). The idea is that a quantified argument QA should govern the minimal sentence that contains all of QA's anaphors, but if that sentence contains other quantified arguments that have their own anaphors, that minimal sentence should contain these anaphors as well.

The Quarc incorporates one last feature of natural language that has no parallel in the Predicate Calculus. Let us compare the following two sentences:

39. John loves Mary
40. Mary is loved by John

In both we say that the same relation holds in the same way between the same two individuals, while changing the order in which these individuals are mentioned in the sentence. We can do this because the predicate 'love' has an active as well as a passive form, and changing the order of names together with a change from active to passive does not affect what is said. All natural languages have the means of similarly reordering the arguments of a many-place predicate in any possible way, although this need not be accomplished by different predicate forms: some languages do it by means of cases and some by means of prepositions, for instance.

Although such a reorder does not affect any change of meaning as long as we limit ourselves to non-quantified arguments, once we allow quantified ones a change of meaning might result:

41. Every man loves some women
42. Some women are loved by every man

Incorporating quantification in the Quarc the way it is in natural language, we should also have the means to realise such a reordering. We shall do that by having, for any n-place predicate, n! reordered forms, indicated by the order of superscripts. For instance, if P is a two-place predicate, it will have the reordered form $P^{2,1}$; if it is a three-place predicate, it will have the five reordered forms $P^{1,3,2}$, $P^{2,1,3}$, $P^{2,3,1}$, $P^{3,1,2}$ and $P^{3,2,1}$. Wherever convenient, I shall consider P itself as P under the identity reorder and write $P^{1,2}$ if it is a two-place predicate, etc. Accordingly, sentences (39) to (42) will be translated, respectively, as

43. $(j,m)L$
44. $(m,j)L^{2,1}$
45. $(\forall M,\exists W)L$
46. $(\exists W,\forall M)L^{2,1}$

Sentences (42) and (46) should be distinguished from the following:

47. Some women love every man
48. $(\exists W,\forall M)L$

The relation between the meaning of a predicate and that of its reordered forms will be specified by means of our derivation rules and rules for truth-value assignments.[10]

III. The Formal System[11]

The Language

Our language has the following symbols:

- Predicates: P, Q, R …, to each of which we assign a natural number, called its *number of places*.
- Singular arguments (SAs): a, b, c …
- Anaphors: α, β, γ …
- Sentential operators: $\neg$, $\wedge$, $\vee$, $\rightarrow$.
- Quantifiers: $\forall$, $\exists$.
- Numerals used as indices, comma, parentheses.

If P is a one-place predicate, then ∀P and ∃P will be called *quantified arguments* (QAs). An *argument* is either a singular argument or a quantified one. Anaphors are not considered arguments.

An occurrence of an anaphor α is anaphoric on an argument A if A is written to the left of α, α is written as a subscript to the right of A (A_α), and α is not written as a subscript to the right of any argument occurring between it and A. I shall call A the *source* of α. Notice that A is either singular or quantified. In A_α, A is considered as the argument, and not A_α.

If P is an n-place predicate, then P with any permutation of the numbers 1 to n written as superscripts separated by commas to its right is called a *reordered* form of P. For example, if P is a three-place predicate, then $P^{3,1,2}$ is a reordered form of P. P itself will occasionally be considered as the *identity* reordered form of P ($P^{1,2,3}$ in this example).

Formulas

The following rules specify all the ways in which formulas can be generated.

Definition 1.1 (Basic Formulas). If P is an n-place predicate (n≥1) and a_1, a_2, … a_n singular arguments (not necessarily different), then $(a_1, a_2, \dots a_n)$P is a formula, called a *basic* formula.

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[10] It is often claimed that sentences like (41), (42) and (47) are ambiguous. Although such ambiguity occasionally exists, I think that its prevalence has been exaggerated; moreover, even in cases in which it does exist, the sentences usually have a default reading (adopted here), which is rejected only in case it does not make sense in the context. In rejecting ambiguity from the formal system I introduce a measure of regimentation into the logic (as is also done by Moss), typical of such systems.

Francez, by contrast, incorporates the alleged ambiguity in his system (Example 2.2), which makes him ignore natural language's need of reordering devices such as active–passive forms with their logical relations, and also complicates his system with issues of scope. In the incorporation of reordering devices, my logical syntax is again closer to that of natural language than is his. This incorporation is mentioned by Moss as something for 'future work' (2010: 563).

As was mentioned above, Francez continues to use standard variables. I think this also introduces a measure of unnecessary complexity to his syntax. For instance, while I formalise 'Some men love every woman' as (∃M,∀W)L, Francez would formalise it $L(\exists x.M(x)_1,\forall y.W(y)_2)$; see his Example 2.2.

[11] This section is indebted to the work on the formal system by Lanzet, contained in (Lanzet & Ben-Yami 2004) and (Lanzet 2006).

Definition 1.2 (Reorder). If P is a reordered n-place predicate ($n>1$) and a_1, a_2, … a_n singular arguments, then $(a_1, a_2, \dots a_n)P$ is a formula.

Definition 1.3 (Negative Predication). If P is an n-place predicate or a reordered n-place predicate ($n\geq 1$) and a_1, a_2, … a_n singular arguments, then $(a_1, a_2, \dots a_n)\neg P$ is a formula.

Definition 1.4 (Sentential Operators). If φ and ψ are formulas, then so are $\neg(\varphi)$, $(\varphi)\wedge(\psi)$, $(\varphi)\vee(\psi)$ and $(\varphi)\rightarrow(\psi)$.

Definition 1.5 (Anaphora). If φ is a formula containing, from left to right, occurrences a_1 to a_n of the singular argument a ($n>1$), none of which is the source of any anaphor, and α does not occur in φ, then $\varphi(a_\alpha/a_1, \alpha/a_2, \dots \alpha/a_n)$ is also a formula.[12]

Before we define how to generate formulas that contain quantified arguments, we need to define when a QA *governs* a string of symbols (which does not have to be a formula):

Definition 2 (Governance). An occurrence QA of a quantified argument governs a string of symbols φ just in case QA is the leftmost quantified argument in φ, and φ does not contain any other string of symbols (ψ) in which the parenthesis are a pair of sentential parenthesis and ψ contains QA and all the anaphors of all quantified arguments occurring in it (in ψ).

A pair of sentential parenthesis are those introduced according to Definition 1.4 around a formula, not those surrounding arguments. Whether φ contains such a string (ψ) is determined before omission of parentheses. According to this definition, $\exists P$ governs the strings $\exists PQ$, $(a,\exists P)R$ and $(\exists P,\forall S)R$ (that these are also formulas will be clear from the following definition). $\exists P$ does not govern the string $(\forall S,\exists P)R$, for it is not the leftmost quantified argument in it. Nor does it govern the string $(\exists PQ)\rightarrow(aQ)$, for that string contains the string $(\exists PQ)$, which contains $\exists P$ and all anaphors of all quantified arguments occurring in it, as there are none. It does however govern the string $(\exists P_\alpha Q)\rightarrow(\alpha Q)$, for here $(\exists P_\alpha Q)$ does not contain the anaphor α of $\exists P_\alpha$. It also governs the formula $((\exists P,\forall S_\alpha)R)\rightarrow(\alpha Q)$, for now the string $((\exists P,\forall S_\alpha)R)$ does not contain the anaphor α of the quantified argument $\forall S$. However, $\exists P$ does *not* govern the formula $((\exists P,a_\alpha)R)\rightarrow(\alpha Q)$: although the string $((\exists P,a_\alpha)R)$ does not contain the anaphor α of the *singular* argument a, it does contain all anaphors of all *quantified* arguments occurring in it.

We can now give the last definition of how formulas are generated:

Definition 1.6 (Quantification). If φ is a formula containing an occurrence a_1 of a singular argument, and if substituting the quantified argument QA for a_1 will result in this occurrence of QA governing φ, then $\varphi(QA/a_1)$ is a formula.

Notice that I wrote that $\varphi(QA/a_1)$ is a formula *if* as a result of the substitution of that occurrence of QA governs the string of symbols, but I did not write that it is a formula *only* if QA governs $\varphi(QA/a_1)$. Consider the formula $\neg(aQ)$. If we substitute $\forall P$ for a,

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[12] $\varphi(a/b)$ is the formula in which a replaced b, for instance, if φ is bP then $\varphi(a/b)$ is aP, and if φ is (a,b)P then $\varphi(a/b)$ is (a,a)P. In case several occurrences of a are substituted by other terms, they are designated a_1, a_2, etc. For instance, if φ is (a,a)P and the occurrences of a are designated, from left to right, a_1 and a_2, then $\varphi(a_\alpha/a_1, \alpha/a_2)$ is $(a_\alpha,\alpha)P$.

then $\forall P$ will not govern the string $\neg(\forall PQ)$, for this string contains the string $(\forall PQ)$, which contains $\forall P$ and no argument in it has anaphors. So $\neg(\forall PQ)$ cannot be generated according to Definition 1.6. Still, since $\forall PQ$ can be generated according to Definition 1.6 from aQ, and $\neg(\forall PQ)$ can be generated from $\forall PQ$ according to Definition 1.4, $\neg(\forall PQ)$ is a formula.

This concludes the definitions of formulas of the language.

Truth-value Assignments

In (Lanzet and Ben-Yami 2004) and (Lanzet 2006) a model-theoretic semantics is provided for calculi close to the one developed above. Here I shall instead develop a truth-valuational substitutional approach to the Quarc. I think the truth-valuational approach has several characteristics that make it of logical interest.[13] As a model-theoretic semantics for the calculus was developed in other publications, it is worthwhile to develop the truth-valuational approach here.

A truth-value assignment assigns a unique truth-value, either *true* or *false*, to each formula of the Quarc, according to the following rules:[14]

Definition 3, Truth-value Assignment

Basic Formulas. Every basic formula is assigned either *true* or *false*, but not both.

Reorder. Let P be an n-place predicate (n>1) and $\pi = \pi 1, \pi 2, \dots \pi n$ a permutation of 1, 2, … n. The truth-value assigned to $(a_{\pi 1}, a_{\pi 2}, \dots a_{\pi n})P^{\pi}$ is that assigned to $(a_1, a_2, \dots a_n)P$.

Sentential Operators. Let φ and ψ be formulas. If φ is true then $\neg(\varphi)$ is false; if φ is false then $\neg(\varphi)$ is true. If φ and ψ are true, so is $(\varphi)\wedge(\psi)$; otherwise it is false. Etc.

Negative Predication. Let P be an n-place predicate or a reordered n-place predicate ($n\geq 1$) and $a_1, a_2, \dots a_n$ singular arguments. The truth-value assigned to $(a_1, a_2, \dots a_n)\neg P$ is that assigned to $\neg(a_1, a_2, \dots a_n)P$.

Anaphora. If φ is a formula containing, from left to right, occurrences a_1 to a_n of the singular argument a (n>1), none of which has any anaphors, and α does not occur in φ, then the truth-value assigned to $\varphi(a_\alpha/a_1, \alpha/a_2, \dots \alpha/a_n)$ is that assigned to φ.

Particular Quantification. Let $\varphi(\exists P)$ be a formula governed by an occurrence of $\exists P$. If there is some singular argument a for which aP is true and $\varphi(a/\exists P)$ is true, then $\varphi(\exists P)$ is true; if for every singular argument for which aP is true $\varphi(a/\exists P)$ is false, then $\varphi(\exists P)$ is false.

Universal Quantification. Let $\varphi(\forall P)$ be a formula governed by an occurrence of $\forall P$. If for every singular argument a for which aP is true $\varphi(a/\forall P)$ is also true, then $\varphi(\forall P)$ is true; if for some singular argument for which aP is true $\varphi(a/\forall P)$ is false, then $\varphi(\forall P)$ is false.

Instantiation. For every one-place predicate P there is some singular argument a for which aP is assigned *true*.

[13] See (Ben-Yami, unpublished).

[14] It can be shown that these rules determine for any formula on any assignment a unique truth-value.

The idea behind Instantiation is in need of explanation. It derives from the presupposition involved when a quantified phrase is used as an argument of a predicate. The use of the sentences 'Some men are philosophers', 'All men are mortal', 'Seven men are in the room', 'Most men have arrived' and even 'No man is immortal' presupposes that there are men. More generally, a sentence of the form 'qP are R' presupposes that P has instances.[15] Since we allow any one-place predicate to occur, quantified, in argument position, we should accordingly take care that any one-place predicate has instances. The truth-value assignment rules for the quantifiers were formulated with this presupposition in view, deliberately leaving unspecified the truth-value of $\varphi(\exists P)$ and $\varphi(\forall P)$ in case 'aP' is false for all 'a'.

We could of course allow truth-value assignments on which for some P, aP may not be true for any a. A model-theoretic version of this is developed in (Lanzet 2006), where P is allowed to have an empty extension and $\varphi(\exists P)$ and $\varphi(\forall P)$ are then defined as neither true nor false. This can be considered a Quarc analogue of free logic: since in the Quarc a one-place predicate P can occur, quantified, in argument position, it has in this respect a role similar to that of singular arguments or individual constants, and it is therefore assumed in the basic model-theoretic treatment not to be empty. The elimination of this assumption is therefore similar to the elimination of the assumption that individual constants are not empty. Although these Quarc analogues of free logic may be of interest, I do not pursue them in this paper: here I present the basic version of the calculus, which should serve as the point of departure for any more elaborate version.

Validity on the truth-valuational substitutional approach is defined while allowing the addition and elimination of individual constants—the singular arguments of Quarc—to and from the language; namely, validity is independent of a specific individual constant list of the language.[16] Accordingly, an argument whose premises are the formulas Ψ and whose conclusion is φ is valid just in case every assignment of truth-values that makes all formulas Ψ true makes φ true as well, even if we add or eliminate singular arguments to or from our language (only singular arguments not occurring in Ψ and φ can of course be eliminated). We then write, $\Psi \vDash \varphi$. In case a formula φ is true on all assignments, even if we add or eliminate singular arguments from our language, φ is said to be valid or a logical truth and we write $\vDash \varphi$.

Derivation Rules

My method of writing proofs in Natural Deduction is based on the system introduced by Jaśkowski (1934) and further developed and streamlined by Fitch (1954) and others. A proof is a sequence of lines written according to the derivation rules specified below. Every line in the proof contains its line number in parentheses, to its right a formula, to its left a (possibly empty) list of line numbers of formulas on which the formula to its right is said to depend, and to the right of the formula the justification of this occurrence of the formula. The justification usually consists of a name of a derivation rule and the numbers of lines to which the rule is applied. Here is an elementary example:

1	(1)	$aP \wedge bQ$	Premise
1	(2)	aP	$\wedge$-Elimination 1

[15] See (Strawson 1950, § V.c), (Strawson 1952, § 6.III.7) and (Ben-Yami 2004, passim).

In earlier publications I called this rule Referential Import, claiming that the sentence '(qP)R' presupposes *reference* to Ps. Since I adopt in this paper a truth-valuational substitutional approach to logic, which does not use the concept of reference, I prefer to talk here of P having *instances*, which can be interpreted on this approach as meaning that some sentence of the form 'aP' is true.

[16] See (Dunn and Belnap 1968: 183), (LeBlanc 1983: 190), (Ben-Yami, unpublished: § III).

The numbers in the list of line numbers to the left shall be written without repetitions and in ascending order.

The formula in the last line is the *conclusion* of the proof, and it is said to be proved from the formulas on which it depends. If a formula φ is provable from formulas $\psi_1, \dots \psi_n$, we write $\psi_1, \dots \psi_n \vdash \varphi$. If Ψ are formulas, possibly infinitely many, and there is a proof of φ from some or all of Ψ, we say that φ is provable from Ψ and write $\Psi \vdash \varphi$. As we shall see, a conclusion depends only on premises. In case the list of line numbers of formulas on which the conclusion depends is empty (namely, there is no list to the left of the line number of the conclusion), the conclusion is said to be a theorem and we write $\vdash \varphi$.

I proceed to list the derivation rules of the Quarc.

Definition 4 (Derivation Rules)

Definition 4.1 (Premise). At any stage in a proof any formula can be written, depending on itself, its justification being Premise:

i	(i)	φ	Premise

Definition 4.2 (Propositional Calculus Rules, PCR). We allow the usual derivation rules of the Propositional Calculus (see the example above), but with a constraint that I shall now explain.

Consider for instance the formula $\varphi\rightarrow\psi$: if ψ contains an anaphor whose source is in φ, then the apparent modus ponens inference from φ and $\varphi\rightarrow\psi$ to ψ should be blocked.[17] This is done by demanding, for each rule, that all the relevant symbols in its formulation stand for formulas. For instance, for the rule $\varphi, \varphi\rightarrow\psi \vdash \psi$ we demand that φ and ψ be formulas. This makes it impossible for ψ to contain an anaphor whose source is in φ.

Definition 4.3 (Sentence-negation to Predication-negation, SP). Let P be an n-place predicate or a reordered n-place predicate ($n\geq1$), and $a_1, a_2, \dots a_n$ singular arguments. Suppose we are given a proof with j-1 lines containing the line (i) as below (where L is a list, possibly empty, of line numbers); we can then add to the proof a line (j) as follows:

L	(i)	$\neg((a_1, a_2, \dots a_n)P)$	
L	(j)	$(a_1, a_2, \dots a_n)\neg P$	SP i

The justification of line (i) has been omitted in the formulation because it is irrelevant to the rule.

---

[17] Otherwise we could infer, say, from $(\forall M_\alpha,\alpha)L \rightarrow \alpha N$ and $(\forall M_\alpha,\alpha)L$ the string αN, which is not even a formula (these sentences can be seen as translating, respectively, 'If a man loves himself then he is a narcissist', 'Every man loves himself' and 'He is a narcissist').

Another example, due to Lanzet, of an invalid inference which should be blocked and in which a formula is derived by means of conjunction elimination, is the one from $(\exists M_\alpha H\wedge\alpha O \rightarrow \forall SC)\wedge\alpha G$ to $(\exists M_\alpha H\wedge\alpha O \rightarrow \forall SC)$. It can be seen as translating, respectively, 'Some man, if he is in the headquarters and he gives an order, then all soldiers charge, and he is a general', and 'If some man is in the headquarters and he gives an order, then all soldiers charge'.

We shall see later in the paper that this and the following rule generalise to other operators that operate both as sentential operators and as predication operators.

Definition 4.4 (Predication-negation to Sentence-negation, PS). Let P be an n-place predicate or a reordered n-place predicate ($n \geq 1$), and $a_1, a_2, \ldots a_n$ singular arguments.

L	(i)	$(a_1, a_2, \ldots a_n)\neg P$	
L	(j)	$\neg((a_1, a_2, \ldots a_n)P)$	PS i

Definition 4.5 (Reorder, R). Let P be an n-place predicate ($n>1$), $a_1, a_2, \ldots a_n$ singular arguments, and $\pi = \pi 1, \pi 2, \ldots \pi n$ and $\varrho = \varrho 1, \varrho 2, \ldots \varrho n$ two permutations of 1, 2, … n (the identity permutation included).

L	(j)	$(a_{\pi 1}, a_{\pi 2}, \ldots a_{\pi n})P^{\pi}$	
L	(j)	$(a_{\varrho 1}, a_{\varrho 2}, \ldots a_{\varrho n})P^{\varrho}$	R i

Definition 4.6 (Anaphora Introduction, AI). Let φ be a formula containing, from left to right, occurrences a_1 to a_n of the singular argument a ($n>1$), none of which has any anaphors, and suppose α does not occur in φ.

L	(i)	φ	
L	(j)	$\varphi(a_\alpha/a_1, \alpha/a_2, \ldots \alpha/a_n)$	AI i

Definition 4.7 (Anaphora Elimination, AE). Let φ be a formula containing, from left to right, occurrences a_1 to a_n of the singular argument a ($n>1$), none of which has any anaphors, and suppose α does not occur in φ.

L	(i)	$\varphi(a_\alpha/a_1, \alpha/a_2, \ldots \alpha/a_n)$	
L	(j)	φ	AE i

Definition 4.8 (Universal Elimination, UE). Let $\varphi(\forall P)$ be a formula governed by an occurrence of $\forall P$.

L1	(i)	$\varphi(\forall P)$	
L2	(j)	aP	
L1∪L2	(k)	$\varphi(a/\forall P)$	UE i, j

L1∪L2 is the number list whose numbers occur either in L1 or L2.

Definition 4.9 (Universal Introduction, UI). Let $\varphi(\forall P)$ be a formula governed by an occurrence of $\forall P$. Assume that neither $\varphi(\forall P)$ nor the formulas in lines L apart from aP in line (i) contain any occurrence of the singular argument a.

i	(i)	aP	Premise
L	(j)	$\varphi(a/\forall P)$	
L–i	(k)	$\varphi(\forall P)$	UI i, j

L–i is the (possibly empty) number list whose numbers are all those in L apart from i.

Definition 4.10 (Particular Introduction, PI). Let φ(∃P) be a formula governed by an occurrence of ∃P.

L1	(i)	φ(a/∃P)	
L2	(j)	aP	
L1∪L2	(k)	φ(∃P)	PI i, j

Definition 4.11 (Instantiation, INS). Let q stand for either ∃ or ∀, and φ(qP) be a formula governed by an occurrence of qP. Assume that the singular argument a does not occur in any of the formulas in lines L1 or L2–j–k and also does not occur in φ(qP) or ψ.

L1	(i)	φ(qP)	
j	(j)	aP	Premise
k	(k)	φ(a/qP)	Premise
L2	(l)	ψ	
L1∪L2–j–k	(m)	ψ	INS i, j, k, l

I believe most rules are quite intuitive, and they will be even more so once we will have seen a few examples. The last rule, Instantiation, relies among other things on the requirement that a formula of the form φ(qP) has to have true instances for it to be true, a requirement which as we have seen is built into the truth-value assignment rules for quantifiers. Informally, we can capture the idea behind this rule as follows: if a quantified formula φ(qP) is true, then so is some formula or other of the form aP and φ's substitution instance, φ(a/qP); accordingly, if some formula ψ follows from these latter formulas then it follows from the former one as well. That this rule preserves soundness will be proved below.

Examples of Proofs

I start by proving that 'All men are mortal' entails 'Some men are mortal', or formally and generally:

(∀P)Q ⊢ (∃P)Q

1	(1)	∀PQ	Premise
2	(2)	aQ	Premise
3	(3)	aP	Premise
2, 3	(4)	∃PQ	PI 2, 3
1	(5)	∃PQ	INS 1, 3, 2, 4

Let us next prove that 'Some men aren't Greek' entails the negation of 'All men are Greek', namely:

(∃P)¬Q ⊢ ¬((∀P)Q)

1	(1)	∃P¬Q	Premise

2	(2)	$\forall PQ$	Premise
3	(3)	aP	Premise
4	(4)	$a\neg Q$	Premise
4	(5)	$\neg aQ$	PS 4
2, 3	(6)	aQ	UE 2, 3
3, 4	(7)	$\neg\forall PQ$	PCR (Negation Introduction) 2, 6, 5
1	(8)	$\neg\forall PQ$	INS 1, 3, 4, 7

These two inferences are included in the traditional Square of Opposition; all other logical relations of the Square can also be proved in the Quarc.

I next prove that 'Some men are Greek' entails 'Some Greeks are men':

$(\exists P)Q \vdash (\exists Q)P$

1	(1)	$\exists PQ$	Premise
2	(2)	aP	Premise
3	(3)	aQ	Premise
2, 3	(4)	$\exists QP$	PI 2, 3
1	(5)	$\exists QP$	INS 1, 2, 3, 4

This inference is one of those considered *immediate* in Aristotelian logic. All other immediate inferences that can be formalised in the Quarc can also be proved in it.

Let us now prove that 'All Greek are men' and 'All men are mortal' entails 'All Greek are mortal', the traditional Barbara:

$(\forall P)Q, (\forall Q)R \vdash (\forall P)R$

1	(1)	$\forall PQ$	Premise
2	(2)	$\forall QR$	Premise
3	(3)	aP	Premise
1, 3	(4)	aQ	UE 1, 3
1, 2, 3	(5)	aR	UE 2, 4
1, 2	(6)	$\forall PR$	UI 3, 5

All other Aristotelian syllogisms can also be proved in the Quarc. Accordingly, all the inferences of Aristotelian logic that can be formalised in the Quarc are provable in it. This, I think, is a desirable result for a system purporting to provide an improved representation of the logic of natural language. The fact that validity is not generally preserved under the standard translation of those inferences into the Predicate Calculus may indicate that the logic of the latter diverges from that of the former.[18]

The following is an example of a theorem of the system, 'All men are men':

$\vdash (\forall P)P$

1	(1)	aP	Premise
	(2)	$\forall PP$	UI, 1, 1

[18] For more on the relation of Aristotelian logic to the Logic of Quantified Arguments, see (Ben-Yami 2004, Chap. 10).

We turn to inferences involving multiply quantified sentences. Let us first prove that 'Every man loves every man' entails 'Every man loves himself'.

$(\forall P,\forall P)R \vdash (\forall P_\alpha,\alpha)R$

1	(1)	$(\forall P,\forall P)R$	Premise
2	(2)	aP	Premise
1, 2	(3)	$(a,\forall P)R$	UE 1, 2
1, 2	(4)	$(a,a)R$	UE 3, 2
1, 2	(5)	$(a_\alpha,\alpha)R$	AI 4
1	(6)	$(\forall P_\alpha,\alpha)R$	UI 2, 5

Next, 'Every man loves every woman' entails 'Every woman is loved by every man':

$(\forall P,\forall Q)R \vdash (\forall Q,\forall P)R^{2,1}$

1	(1)	$(\forall P,\forall Q)R$	Premise
2	(2)	aP	Premise
3	(3)	bQ	Premise
1, 2	(4)	$(a,\forall Q)R$	UE 1, 2
1, 2, 3	(5)	$(a,b)R$	UE 4, 3
1, 2, 3	(6)	$(b,a)R^{2,1}$	R 5
1, 3	(7)	$(b,\forall P)R^{2,1}$	UI 2, 6
1	(8)	$(\forall Q,\forall P)R^{2,1}$	UI 3, 7

Also, 'Some woman is loved by every man' entails 'Every man loves some woman':

$(\exists Q,\forall P)R^{2,1} \vdash (\forall P,\exists Q)R$

1	(1)	$(\exists Q,\forall P)R^{2,1}$	Premise
2	(2)	aP	Premise
3	(3)	bQ	Premise
4	(4)	$(b,\forall P)R^{2,1}$	Premise
2, 4	(5)	$(b,a)R^{2,1}$	UE 4, 2
2, 4	(6)	$(a,b)R$	R 5
2, 3, 4	(7)	$(a,\exists Q)R$	PI 6, 3
3, 4	(8)	$(\forall P,\exists Q)R$	UI 2, 7
1	(8)	$(\forall P,\exists Q)R$	INS 1, 3, 4, 8

To demonstrate the use of anaphors in donkey sentences, consider first the following inference: 'If John likes a donkey, he buys it', 'If John buys a donkey, he vaccinates it', so 'If John likes a donkey, he vaccinates it':

$(a_\alpha,\forall P_\beta)Q\rightarrow(\alpha,\beta)R,\ (a_\alpha,\forall P_\beta)R\rightarrow(\alpha,\beta)S \vdash (a_\alpha,\forall P_\beta)Q\rightarrow(\alpha,\beta)S$

1	(1)	$(a_\alpha,\forall P_\beta)Q\rightarrow(\alpha,\beta)R$	Premise
2	(2)	$(a_\alpha,\forall P_\beta)R\rightarrow(\alpha,\beta)S$	Premise

3	(3)	bP	Premise
1, 3	(4)	$(a_\alpha,b_\beta)Q\rightarrow(\alpha,\beta)R$	UE 1, 3
1, 3	(5)	$(a,b_\beta)Q\rightarrow(a,\beta)R$	AE 4
1, 3	(6)	$(a,b)Q\rightarrow(a,b)R$	AE 5
2, 3	(7)	$(a_\alpha,b_\beta)R\rightarrow(\alpha,\beta)S$	UE 2, 3
2, 3	(8)	$(a,b_\beta)R\rightarrow(a,\beta)S$	AE 7
2, 3	(9)	$(a,b)R\rightarrow(a,b)S$	AE 8
1, 2, 3	(10)	$(a,b)Q\rightarrow(a,b)S$	PCR 6, 9
1, 2, 3	(11)	$(a,b_\beta)Q\rightarrow(a,\beta)S$	AI 10
1, 2, 3	(12)	$(a_\alpha,b_\beta)Q\rightarrow(\alpha,\beta)S$	AI 11
1, 2	(13)	$(a_\alpha,\forall P_\beta)Q\rightarrow(\alpha,\beta)S$	UI 3, 12

I now prove that every anti-symmetric relation is irreflexive:

$$(\forall P_\alpha,\forall P_\beta)R\rightarrow\neg(\beta,\alpha)R \vdash (\forall P_\alpha,\alpha)\neg R$$

1	(1)	$(\forall P_\alpha,\forall P_\beta)R\rightarrow\neg(\beta,\alpha)R$	Premise
2	(2)	aP	Premise
1, 2	(3)	$(a_\alpha,\forall P_\beta)R\rightarrow\neg(\beta,\alpha)R$	UE 1, 2
1, 2	(4)	$(a_\alpha,a_\beta)R\rightarrow\neg(\beta,\alpha)R$	UE 3, 2
1, 2	(5)	$(a,a_\beta)R\rightarrow\neg(\beta,a)R$	AE 4
1, 2	(6)	$(a,a)R\rightarrow\neg(a,a)R$	AE 5
1, 2	(7)	$\neg(a,a)R$	PCR 6
1, 2	(8)	$(a,a)\neg R$	SP 7
1, 2	(9)	$(a_\alpha,\alpha)\neg R$	AI 8
1	(10)	$(\forall P_\alpha,\alpha)\neg R$	UI 2, 9

The examples above demonstrate the working and power of the system.

Soundness

As the proof of soundness is quite straightforward, I supply it here only in part.

Theorem 5 (Soundness). If $\Psi \vdash \varphi$ then $\Psi \vDash \varphi$. Namely, if a formula φ is provable from formulas Ψ, then the argument with Ψ as premises and φ as conclusion is valid.

Proof. The proof is in induction on proof length. We show that for any natural number n, for any proof with n lines, if the premises on which the formula in line n depends are true on an assignment, then so is that formula, independently of the singular argument list of the language (the fact that the argument list can be changed plays a role in the completeness proof but not in the soundness one).

The inductive base is a proof of one line. The only rule that enables us to write a first line is Premise, and therefore any one-line proof is of the form

1	(1)	φ	Premise

φ then depends on itself, and since if it is true on an assignment it is true on that assignment, the inductive basis is proved.

We now assume that for any proof of at most n lines, if the premises on which a formula in any of the lines depends are true on an assignment, that formula is also true on that assignment. We prove that this holds for the n+1 line. The n+1 line is written according to one of our eleven derivation rules, so we have to check all eleven cases. I do that here for the SP, UE and INS derivation rules.

If the n+1 line is written according to the Sentence-negation to Predication-negation rule, SP, it looks as follows:

L	(i)	$\neg((a_1, a_2, \ldots a_m)P)$	
L	(n+1)	$(a_1, a_2, \ldots a_m)\neg P$	SP i

Assume that the formulas in lines L, on which formula (n+1) depends, are true on an assignment A. Since these are the formulas on which the formula in line (i) depends, according to the inductive hypothesis that formula is also true on A. But according to the Negative Predication rule for truth-value assignment, on any assignment the truth-value of $(a_1, a_2, \ldots a_m)\neg P$ is that of $\neg(a_1, a_2, \ldots a_m)P$, so it is also true on A. QED

If the n+1 line is written according to the Universal Elimination rule, UE, then it is of the form $\varphi(a/\forall P)$, with $\forall P$ governing $\varphi(\forall P)$. The proof looks as follows:

L1	(i)	$\varphi(\forall P)$	
L2	(j)	aP	
L1∪L2	(n+1)	$\varphi(a/\forall P)$	UE i, j

Assume the formulas in lines L1∪L2 are true on an assignment A. According to our inductive hypothesis, it follows that the formulas in lines (i) and (j), namely $\varphi(\forall P)$ and aP, are also true on A. But then, according to the Universal Quantification Rule for truth-value assignment, it follows that $\varphi(a/\forall P)$ is also true on A. QED

If the n+1 line is written according to the Instantiation rule, then the proof looks as follows:

L1	(i)	$\varphi(qP)$	
j	(j)	aP	Premise
k	(k)	$\varphi(a/qP)$	Premise
L2	(l)	ψ	
L1∪L2–j–k	(n+1)	ψ	INS i, j, k, l

'q' stand for either ∃ or ∀, and $\varphi(qP)$ is governed by an occurrence of qP; 'a' does not occur in any of the formulas in lines L1 or L2–j–k and also does not occur in $\varphi(qP)$ or ψ. Assume now that the formulas in lines L1∪L2–j–k are true on an assignment A. Since 'a' does not occur in the lines L1, the list L1 does not contain j or k, and therefore all the formulas in lines L1 are true on A. From the inductive hypothesis it now follows that the formula in line (i), namely $\varphi(qP)$, is also true on A. But from the truth-value assignment rules for the universal and the particular quantifier, it follows that for $\varphi(qP)$ to be true, there must be a singular argument 'b' for which bP and $\varphi(b/qP)$ are true on A. We substitute 'b' for 'a' in all the lines of the proof. Since no derivation rule depends on the identity of the constants occurring in the formulas, the proof is still according to the derivation rules and the inductive hypothesis still holds. The formula now in line j, namely bP, is true; since φ(qP) did not contain 'a', the formula in line k is now φ(b/qP), and it is also true; and since 'a' does not occur in any of the formulas in lines L2–j–k, all these formulas also remain true after the substitution; consequently, all the formulas in lines L2 are now true, and according to our inductive hypothesis formula ψ(b/a) is also true on A. But since ψ did not contain 'a', ψ(b/a) is ψ, and it is also true on A. QED

It is similarly proved that all other derivation rules preserve validity, and thus, for any n, if $\psi_1, \dots \psi_n \vdash \varphi$ then $\psi_1, \dots \psi_n \vDash \varphi$. It follows that for any formulas Ψ, whether finite or infinite in number, if $\Psi \vdash \varphi$ then $\Psi \vDash \varphi$, namely that the Quarc is sound.

Completeness

The completeness proof for the Quarc is of some length, and squarely belongs to mathematical logic while this paper is intended for a wider readership. I shall therefore not present it here. Completeness is proved with model-theoretic semantics for the closely related calculi developed in (Lanzet and Ben-Yami 2004) and (Lanzet 2006), and I intend to present this proof for the truth-valuational approach to the Quarc in a separate paper. With both soundness and completeness proved, the Quarc is shown to be adequate.

As could be seen in the last two sections, the Quarc manages to translate the family of sentences to which the Predicate Calculus offers translations. It also contains Aristotelian logic and is closer than the Predicate Calculus to the logic of natural language in other respects as well, some witnessed in this paper (modes of predication, quantified arguments, active–passive form distinctions) and some discussed in other works referred to above. The system is also sound and complete. All these features make it an important tool for the study of the logic of natural language.

IV. The Modal Quantified Argument Calculus

When we turn to add modal operators to the Quarc, we should note that, like negation, natural language uses modal operators both as sentential operators and as predication modifiers. We thus have the two forms,

49. John could be a philosopher
50. It is possible that John be a philosopher

And similarly,

51. 2 is necessarily even
52. It is necessarily the case that 2 is even

I shall accordingly introduce the modal operators into the Quarc both as sentential operators and as *predication operators*. The sentences above will be translated in the system as

53. a◊P
54. ◊aP

55. $a\Box P$
56. $\Box aP$

As with negation, if the arguments are singular the sentential and predicative forms will be equivalent, in the sense that one will entail the other and that on any assignment they will have the same truth-value; but that is not generally the case when some of the arguments are quantified. The following two natural language sentences are not synonymous, nor will their translations be equivalent:

57. Necessarily, some people are alive
58. Some people are necessarily alive
59. $\Box\exists SP$
60. $\exists S\Box P$

When a modal operator functions as a sentential operator, I shall call this *de dicto* modality, while when it functions as predication operator I shall call this *de re* modality. As with the modal Predicate Calculus, the modal Quarc or M-Quarc obviously cannot claim to capture with the two modal operators it employs all the modal distinctions natural language makes by means of its rich variety of modal terms. Rather, with our comparatively crude formal modal distinctions we attempt to capture and explore the relations between *some* basic features of modality as incorporated in natural language.

I add formation rules for formulas in which modal operators occur. I first add to the language

- Modal operators: $\Box$, $\Diamond$.

When a modal operator occurs with a formula to its right, I shall say that it functions as a sentential operator. When a modal operator or the negation sign occurs between arguments in parentheses to its left and a predicate letter to its right, I shall say that it functions as a predication operator. When a chain of modal operators and negation signs occurs in this way, I shall also say that they function as predication operators. The two new rules for formulas are:

> **Definition 1.7 (Sentential Modality)**. If φ is a formula, then so are $\Box(\varphi)$ and $\Diamond(\varphi)$.
>
> **Definition 1.8 (Predication Modality)**. If P is an n-place predicate or a reordered n-place predicate ($n\geq1$), $a_1, a_2, \dots a_n$ singular arguments and * a chain (possibly empty) of predication operators, then $(a_1, a_2, \dots a_n)\Box{*}P$ and $(a_1, a_2, \dots a_n)\Diamond{*}P$ are formulas.

Chaining predication operators would allow us to formalize ‘Some philosophers are not necessarily intelligent’ as $\exists S\neg\Box P$, which is not equivalent to $\exists S\Box\neg P$, the formalization of ‘Some philosophers are necessarily not intelligent’. This iteration would also allow formulas such as $a\Box\Box\Diamond P$, whose natural language analogue might be meaningless, as is also the case with the parallel concatenation of sentential operators, $\Box\Box\Diamond\varphi$. This possible meaninglessness of the natural language analogues, if it does not indicate a departure of formal modal logic from the natural idiom (which is a possibility), might be explained by this concatenation being reducible, on some modal systems, to a single operation, and as such redundant and therefore inexistent in natural language.

Definition 1.3 (Negative Predication) should also be changed to allow concatenation of predication operators, $(a_1, a_2, \dots a_n)\neg{*}P$ defined as a formula, where * stands for a chain, possibly empty, of predication operators. This star should also be added to the related rules for truth-value assignments and proofs.

We add the following derivation rules to our system:

Definition 4.12 (Necessitation, NEC). Let φ be any formula.

(i)	φ	
(j)	$\Box\varphi$	NEC i

Notice that φ in line (i) does not depend on any other formula.

According to the modal system that one adopts—K, T, B, S4 and S5 will be considered below—the following axioms should be added to the system:

Definition 4.13 (Modal Axioms). Let φ and ψ be any formulas. The following can be written at any stage in a proof, according to the modal systems adopted:

(i)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	K (any modal system)
(i)	$\Box\varphi \to \varphi$	T (T, B, S4, S5)
(i)	$\varphi \to \Box\Diamond\varphi$	B (B, S5)
(i)	$\Box\varphi \to \Box\Box\varphi$	S4 (S4, S5)
(i)	$\Diamond\varphi \to \Box\Diamond\varphi$	S5 (S5)

If we allow the last S5 axiom, we need not include the axioms B and S4 in S5. Notice that an axiom is written without any list of line numbers of formulas to its left, namely, it depends on no formula—this is what makes it an axiom. I have omitted parentheses in the formulations, for perspicuity; otherwise I should have written, for instance, the T axiom as $(\Box(\varphi)) \to (\varphi)$. We should also either introduce a derivation rule for ◊ or treat it as a defined symbol, abbreviating ¬□¬; as this is standard, I don't elaborate on it in this paper.

Lastly, we generalise the derivation rules SP and PN, so that they cover the relation between the use of any operators as sentential operators and their use as predication operators:

Definition 4.3 (Sentential-operator to Predication-operator, SP). Let P be an n-place predicate or a reordered n-place predicate ($n \geq 1$), $a_1, a_2, \dots a_n$ singular arguments, * a chain (possibly empty) of predication operators and # an operator that can function both as a sentential and as a predication operator. Suppose we are given a proof containing line (i) as below; then we can add to the poof line (j) as follows:

L	(i)	$\#((a_1, a_2, \dots a_n){*}P)$	
L	(j)	$(a_1, a_2, \dots a_n)\#{*}\ P$	SP i

Definition 4.4 (Predication-operator to Sentential-operator, PS). Let P, a_1, a_2, … a_n, * and # be as in Definition 4.3.

L	(i)	$(a_1, a_2, \ldots a_n)\#^* P$	
L	(j)	$\#((a_1, a_2, \ldots a_n)^*P)$	PS i

Although the only predication operators incorporated in the Quarc in this paper are monadic, the same rules can be applied to other operators. We can modify and apply the relevant formula definitions and derivation rules so that they apply to conjunction and disjunction as well, and we then have $aP\wedge aQ \dashv\vdash a(P\wedge Q)$ as well as $aP\vee aQ \dashv\vdash a(P\vee Q)$.[19]

Truth-value assignment rules for formulas containing modal operators should now be provided. First, similarly to the generalisation of Definitions 4.3 and 4.4, we generalise Definition 3 for negative predication as follows.

Definition 3, Truth-value Assignment

Predication Operators. Let P be an n-place predicate or a reordered n-place predicate ($n\geq 1$), a_1, a_2, … a_n singular arguments, * a chain (possibly empty) of predication operators and # an operator that can function both as a sentential and as a predication operator. The truth-value of $(a_1, a_2, \ldots a_n)\#^*P$ is that of $\#(a_1, a_2, \ldots a_n)^*P$.

I proceed to the assignments of truth-values to the formulas $\Box(\varphi)$ and $\Diamond(\varphi)$.[20] With any truth-value assignment A we associate a set of assignments S_A, A's *assignment set*. The following assignment rules hold:

Modal Operators. $\Box(\varphi)$ is true on a truth-value assignment A in case φ is true on every assignment $B\in S_A$; otherwise it is false. $\Diamond(\varphi)$ is true on A in case φ is true on some assignment $B\in S_A$; false otherwise.

Truth-values are assigned recursively as follows. First, on any assignment, we assign truth-values to all formulas that do not contain any modal operator. Next, we assign on every assignment truth-values according to the modal operators rule to all formulas of the form $\Box\varphi$ or $\Diamond\varphi$, where φ is a formula to which a truth-value has already been assigned. We then again assign truth-values to any formula which is formed by any rule apart from the rule for sentential modality from formulas to which we have already assigned a truth-value, and which hasn't yet been assigned a truth-value. We repeat this process until every formula has been assigned on any assignment a truth-value.

Notice that assignments A and B may coincide on the truth-values they assign to basic formulas but not on their assignment sets S_A and S_B, in which case they need not coincide on the truth-values they assign to formulas containing modal operators.

[19] Francez introduces similar compounds. Moss does not introduce them, but unlike me in this paper or Francez, he also formalises nouns containing quantified defining clauses, e.g. 'he who kills *a* man' or 'all who respect *all* mammals' (Pratt-Hartmann and Moss 2008). This was done within the framework of the logic of quantified arguments, as part of an adequate formal system, in (Lanzet 2006). The system in the present paper would have to rely, as does the Predicate Calculus, on paraphrases of such nouns. Moss, unlike (Lanzet 2006), does not distinguish in his formal system between the syntax of quantified subject–predicate sentence and that of a noun with a quantified defining clause. This is a departure from the syntax of natural language, I believe, which uses only for the latter relative pronouns.

[20] The approach developed in this paragraph is very close to that suggested in (LeBlanc 1973, p. 11).

On these rules, $\Diamond\varphi$ is equivalent with $\neg\Box\neg\varphi$: on any assignment A, $\neg\Box\neg\varphi$ is true just in case $\Box\neg\varphi$ is false. Thus, on any assignment A, $\neg\Box\neg\varphi$ is true just in case on some assignment $B \in S_A$ $\neg\varphi$ false. So, on any assignment A, $\neg\Box\neg\varphi$ is true just in case on some assignment $B \in S_A$, φ is true. Namely, on any assignment, $\neg\Box\neg\varphi$ is true just in case $\Diamond\varphi$ is.

Definition 3 for Modal Operators guarantees that for any formulas φ and ψ axiom K is true on any assignment. The other different modal systems are captured by imposing different constraints on assignment sets. I list below the constraints on assignment sets needed to make the different modal axioms valid for any formula φ, namely true on all truth-value assignments. I then prove, as an example, the constraint for the T axiom.

- T: $\Box\varphi \rightarrow \varphi$ $\quad$ $A \in S_A$ (Reflexivity. This constraint corresponds to the accessibility relation between worlds in possible world semantics being reflexive; *mutatis mutandis* below.)
- B: $\varphi \rightarrow \Box\Diamond\varphi$ $\quad$ If $B \in S_A$ then $A \in S_B$ (Symmetry)
- S4: $\Box\varphi \rightarrow \Box\Box\varphi$ $\quad$ If $B \in S_A$ and $C \in S_B$ then $C \in S_A$ (Transitivity)
- S5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ $\quad$ If $B \in S_A$ and $C \in S_A$ then $C \in S_B$ (Euclidean Relation)

I now prove that the T-axiom, $\Box\varphi \rightarrow \varphi$, is true on every assignment A for every formula φ just in case $A \in S_A$. First, the sufficiency of the constraint: if $A \in S_A$, then for any φ, since if $\Box\varphi$ is true on A then φ is true on all assignments in S_A, φ is then true on A as well, and so is $\Box\varphi \rightarrow \varphi$. Secondly, to prove that the constraint is necessary, suppose A is allowed not to belong to S_A. Let us look at an assignment A that gives the formula aP the truth-value F and whose assignment set consists of a single assignment B that gives aP the truth-value T. On the assignment A, $\Box$aP is true while aP is false, and therefore $\Box$aP $\rightarrow$ aP, which is an instance of $\Box\varphi \rightarrow \varphi$, is also false, and thus that T-axiom is invalid. QED

It is easy to see that all other new derivation rules also preserve truth on any assignment, and that the M-Quarc is therefore sound. It can also be proved that each modal system is complete. Again, I leave the proof of completeness to a separate work.

The Barcan Formulas

The following formulas, named after Ruth Barcan, who was the first to study them (1946), are valid and can be proved in Simple Modal Predicate Calculus (see for instance (Menzel 2012)):

The Barcan Formulas

61. $\Diamond\exists x\varphi(x) \rightarrow \exists x\Diamond\varphi(x)$
62. $\forall x\Box\varphi(x) \rightarrow \Box\forall x\varphi(x)$

The Converse Barcan Formulas

63. $\exists x\Diamond\varphi(x) \rightarrow \Diamond\exists x\varphi(x)$
64. $\Box\forall x\varphi(x) \rightarrow \forall x\Box\varphi(x)$

The only modal axiom which the proofs of the converse formulas use is K, while the proofs of the Barcan formulas use both K and B.

However, if these formulas are intended to capture our ordinary modal idiom, none seems to be acceptable, for the analogues of the Barcan formulas in natural language seem to be invalid. I consider only their existential form, expressed by means of the particular quantifier. First, the natural language analogue of the Barcan formula: let us assume (1) that none of the people actually living could have been over three metre tall—that is genetically determined; and that (2) on the other hand, if some specific mutation occurred then there could be people over three metre in height. So, *it is possible that some people be over three metre tall*, yet *it is not the case that some people could be over three metre tall*. (To have sentences even closer in structure to the formulas of the Predicate Calculus, we can limit our domain to people and then use the sentences 'It is possible that some be over three metre tall' and 'Some could be over three metre tall'.) So the natural language sentence corresponding to the Barcan formula would be false in this case. Secondly, suppose the sex of the individual fish of some fish species is irreversibly determined at some embryonic stage after conception, and that it can be either male or female, the actual sex being determined at random. Then it is true that *some males could be females*, but it is not true that *it is possible that some males be females*. The natural language sentence corresponding to the converse Barcan formula would then be false.

This apparent discrepancy between our understanding of modality in natural language and the results of Simple Modal Predicate Calculus made Kripke devise a system of Modal Predicate Calculus in which the Barcan formulas are invalid and cannot be proved (Kripke 1963). But apart from the additional complexity of his system, he had to banish individual constants from the language of Modal Predicate Calculus, which seems an ad hoc and undesirable measure, as well as modify some derivation rules in a way that might also seem ad hoc (Menzel 2012: § 3.2).

When we turn to the Quantified Argument Calculus, it can be shown that corresponding versions of all the Barcan formulas are straightforwardly invalid. I shall show the invalidity of the Barcan formula and its converse for a version that uses the existential or particular quantifier, and on the truth-valuational approach; it can equally be done for the universal quantifier or with possible world semantics.

Let us first write the corresponding formulas for the M-Quarc.

Barcan Formulas

65. $\Diamond\exists SP \rightarrow \exists S\Diamond P$
66. $\forall S\Box P \rightarrow \Box\forall SP$

Converse Barcan Formulas

67. $\exists S\Diamond P \rightarrow \Diamond\exists SP$
68. $\Box\forall SP \rightarrow \forall S\Box P$

I first construct an assignment on which the Barcan formula (65) is false. Consider a truth-value assignment A for which there is a single constant 'a' such that 'aS' is true, but for that constant 'aP' is false. Suppose now S_A is constituted by A and an assignment B, for which 'aP' is false, 'bS' is true and 'bP' is true. On B, '$\exists SP$' is true, since 'bS' and 'bP' are both true. Accordingly, on A, '$\Diamond\exists SP$' is true. On the other hand, for '$\exists S\Diamond P$' to be true on A, there should be some substitution instance of a constant 'c' for 'S' such that both (i) 'cS' and (ii) 'c$\Diamond$P' are true on A. The only constant for which (i) holds is 'a'. Now, 'a$\Diamond$P' has by definition the same truth-value as '$\Diamond$aP'; but since 'aP' is false on both A and B, which constitute S_A, '$\Diamond$aP' and with it 'a$\Diamond$P' are both false on A, and so '$\exists S\Diamond P$' is false on A. Accordingly, the Barcan formula (65) is false on A, and it is therefore invalid.

Next, I construct an assignment on which the converse Barcan formula (67) is false. On A now only for 'a' is 'aS' true, while 'aP' is false. S_A consists of A and B, on which only for 'b' is 'bS' true, but 'bP' is false, and on which 'aP' is true. '$\exists S\Diamond P$' is now true on A just in case '$a\Diamond P$' and so '$\Diamond aP$' are; but the latter are true, because 'aP' is true on B. So on A, '$\exists S\Diamond P$' is true. For '$\Diamond\exists SP$' to be true on A, '$\exists SP$' should be true on either A or B. But on A it is not true because 'aP' is not true, and on B it is not true because 'bP' is not true. So '$\Diamond\exists SP$' is false on A. Accordingly, the converse Barcan formula (67) is false on A and the formula is invalid.

Since the M-Quarc is sound, it follows that none of the Barcan formulas is provable.

Necessary Existence?

In Simple Modal Predicate Calculus the formula $\Box\exists x(x=a)$, which is often interpreted as the necessity of existence applied to a particular constant a, is valid and can be proved. Its proof proceeds as follows:

(1)	$a=a$	IdI
(2)	$\exists x(x=a)$	PI 1
(3)	$\Box\exists x(x=a)$	NEC 2

Formula (3) is taken to translate sentences such as 'Necessarily, there is someone who is Socrates' or 'Necessarily, someone is Socrates'. Yet such statements seem false: surely it is possible that no one would have been Socrates! This result has consequently occasioned many metaphysical reflections.[21]

By contrast, the formula $\exists x\Box(x=a)$, which can also be proved, is not considered problematic. It is interpreted as claiming that identity holds necessarily; Socrates is necessarily Socrates, which seems correct: Socrates couldn't have been Plato.

The analogue of $\Box\exists x(x=a)$ is not a theorem of the M-Quarc. I have not introduced identity into the formal system developed above, but as I said in the Introduction, this is because its incorporation in the system need not depart from its standard formal treatment. I shall therefore rely here on that familiar treatment. The analogue of $\Box\exists x(x=a)$ in the Quarc involves particular quantification, and this is always done by having the particular quantifier join a predicate to form a quantified argument. The analogue of $\Box\exists x(x=a)$ would thus be

69. $\Box(\exists S=a)$

If we now try to adapt the Predicate Calculus proof into the Quarc, it will fail as follows:

	(1)	$a=a$	IdI
2	(2)	aS	Premise
2	(3)	$\exists S=a$	PI 1, 2
	~~(4)~~	~~$\Box(\exists S=a)$~~	~~NEC 3~~

Line (4) not in accordance with NEC because line (3) depends on previous lines, while NEC requires that it be a theorem. Any other attempt to prove formula (69) will also fail, since the calculus is sound and this formula is not a logical truth. That it is not a logical truth can be shown as follows. Consider the following truth-value assignment A. On A,

[21] Kripke's system of the Modal Predicate Calculus does not contain the formula $\Box\exists x(x=a)$, or even the formula $a=a$, since it does not contain constants. The generalisation $\forall y\Box\exists x(x=y)$ is not provable or valid in this system, but again this is due to the same problematic measures that were mentioned above.

aS is true, and of course a=a is true as well; S_A consists of A and of the assignment B, on which aS is false; so $\exists$S=a is false on B, and accordingly $\Box(\exists S{=}a)$ is false on A.

By contrast, the Quarc analogue of $\exists x\Box(x{=}a)$ can be proved. Assuming that Socrates is a man, it will say that some man is necessarily Socrates:

$aS \vdash (\exists S)\Box{=}a$

1	(1)	aS	Premise
	(2)	a=a	IdI
	(3)	$\Box$a=a	NEC 2
	(4)	a$\Box$=a	SP 3
1	(5)	$\exists$S$\Box$=a	PI 1, 4

The argument can also be shown to be valid.

A theorem of the M-Quarc that is closer in structure to that of necessary existence in the Predicate Calculus, being of the form $\Box\varphi$, is the following one:

$\vdash \Box(\forall M_\alpha{=}\alpha)$

1	(1)	aM	Premise
	(2)	a=a	IdI
	(3)	$a_\alpha{=}\alpha$	AI 2
	(4)	$\forall M_\alpha{=}\alpha$	UI 3, 1
	(5)	$\Box\forall M_\alpha{=}\alpha$	NEC 4

This formula translates into the Quarc sentences like 'Necessarily, every man is himself'. I believe we consider this sentence true, unlike the way we consider 'Necessarily, there is someone who is Socrates' or 'Necessarily, someone is Socrates'. I therefore think this is a desirable result of the M-Quarc. Similar theorems (which are of course also valid), which seem to me unobjectionable in the same way, are $\Box\forall M{=}\exists M$ (Necessarily, every man is some man) and $\Box\exists M_\alpha{=}\alpha$ (Necessarily, some man is himself).

I have argued in the publications referred to above that the Logic of Quantified Arguments offers a better analysis of the logic of natural language than does the Predicate Calculus on any of its available versions; and we have also seen some reasons for maintaining this in earlier sections of this paper. We now see that the modal extension of the Quarc supports the same conclusion. The M-Quarc is manifestly closer in its syntax to the grammar of modality in natural language, and it also represents better than does the Predicate Calculus some logical relations between modal constructions in natural language, as was demonstrated for the case of the Barcan formulas and necessary existence. It therefore seems to be the more appropriate formal tool for the study of these logical relations.[22]

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