Quasilinear Elliptic Equations Involving Variable Exponents

Mihai Mihăilescu[∗,†] and Gheorghe Moroşanu[∗]

[∗]*Department of Mathematics, Central European University, 1051 Budapest, Hungary*
[†]*Department of Mathematics, University of Craiova, 200585 Craiova, Romania*

Abstract. Consider the boundary value problem $-\sum_{i=1}^{N}\partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\,\partial_{x_i}u\right)=\lambda(x)|u|^{q(x)-2}u$ in Ω, $u=0$ on $\partial\Omega$, where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, while p_1, ..., p_N, q are continuous functions and $q(x)>1$, $p_i(x)\geq 2$ for all $x\in\overline{\Omega}$, $i=1,\cdots,N$. Combining the mountain pass theorem of Ambrosetti and Rabinowitz [1] and Ekeland's variational principle [7] we show that under suitable conditions the above problem has two nontrivial weak solutions. We also consider the eigenvalue problem corresponding to the case when λ in the above equation is a positive constant. We assume that there exist j, $k\in\{1,...,N\}$ with $j\neq k$ such that $p_j\equiv q$ in $\overline{\Omega}$, and q is independent of x_j with $\max_{\overline{\Omega}}q<\min_{\overline{\Omega}}p_k$. Then there exist $0<\lambda_0\leq\lambda_1$ such that every $\lambda\in(\lambda_1,\infty)$ is an eigenvalue, while no $\lambda\in(0,\lambda_0)$ can be an eigenvalue.

Keywords: anisotropic Sobolev spaces, mountain pass theorem, Ekeland's variational principle, eigenvalue, critical point.
PACS: 35D05, 35J60, 58E05.

INTRODUCTION

We are concerned with elliptic problems of the form

$$\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}\left(a_i(x,\partial_{x_i}u)\right)=\lambda(x)|u|^{q(x)-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, the functions $a_i(x,t)$ are of the type $|t|^{p_i(x)-2}t$ with $p_i\in C(\overline{\Omega})$, $p_i(x)\geq 2$ for all $x\in\overline{\Omega}$, $i=1,...,N$, $q\in C(\overline{\Omega})$, $\min_{\overline{\Omega}}q>1$, and $\lambda\in L^\infty(\Omega)$. In the particular case when $a_i(x,t)=|t|^{p(x)-2}t$ for all $i\in\{1,...,N\}$, with $p(x)\in C(\overline{\Omega})$ and $\inf_\Omega p>1$ the differential operator in (1) is the $p(\cdot)$–Laplace operator, i.e., $\Delta_{p(\cdot)}u:=\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. Here, we allow different variable exponents for different spacial directions, $a_i(x,t)=|t|^{p_i(x)-2}t$, so that we need an adequate functional framework for our problem, more precisely an anisotropic Sobolev space with variable exponents. In fact, in Theorem 1 below, which addresses existence and multiplicity of nontrivial weak solutions, even more general a_i's are allowed, for instance $a_i(x,t)=(1+t^2)^{(p_i(x)-2)/2}t$ with $p_i\in C(\overline{\Omega})$ and $\inf_\Omega p_i\geq 2$ for all $i\in\{1,...,N\}$.

It is worth mentioning that problem (1) (considered in the isotropic case, i.e. when all the functions a_i are equal) can serve as a model for phenomena which arise from the study of electrorheological fluids (see [3, 9, 14, 16, 19]), image processing (see [2]), or the theory of elasticity (see [22]). Finally, we note that the case $a_i(x,t)=|t|^{m_i-2}t$ with $m_i>1$ positive constants was studied in [8].

MAIN RESULTS

In order to state our results we need to introduce the so-called

Sobolev spaces with variable exponents. Let us recall first the definitions of the spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in R^N. We will also introduce an adequate functional space for problems of type (1).

Set $C_+(\overline{\Omega})=\{h\in C(\overline{\Omega})\,:\,\min_{x\in\overline{\Omega}}h(x)>1\}$. For $h\in C_+(\overline{\Omega})$ we set

$$h^+=\sup_{x\in\Omega}h(x),\ h^-=\inf_{x\in\Omega}h(x).$$

For $p\in C_+(\overline{\Omega})$, we define the so-called *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega)=\{u\,:\,u \text{ is a measurable real–valued function such that } \int_\Omega|u(x)|^{p(x)}\,dx<\infty\},$$

endowed with the *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\},$$

which is a separable and reflexive Banach space. If $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.

Now, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)}.$$

We point out that the above norm is equivalent with the following norm

$$\|u\|_{p(\cdot)} = \sum_{i=1}^N |\partial_{x_i} u|_{p(\cdot)},$$

provided that $p(x) \ge 2$ for all $x \in \overline{\Omega}$ (see [11]). Hence $W_0^{1,p(\cdot)}(\Omega)$ is a separable, reflexive Banach space. Note that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, where $p^\star(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \ge N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact. For details on variable exponent Lebesgue and Sobolev spaces we refer to the book of Musielak [15] and the papers of Kováčik and Rákosník [10], Edmunds et al. [4, 5, 6], Samko and Vakulov [20].

Now, we introduce a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ that will serve as our functional framework for problem (1). Let us denote by $\overline{p} : \overline{\Omega} \to R^N$ the vectorial function $\overline{p} = (p_1, ..., p_N)$. We define $W_0^{1,\overline{p}(\cdot)}(\Omega)$, the *anisotropic Sobolev space with variable exponents*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overline{p}(\cdot)} = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}.$$

As it was pointed out in [13], $W_0^{1,\overline{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

We also note that in the case when p_i are constant functions, the resulting anisotropic Sobolev space is denoted by $W_0^{1,\overline{p}}(\Omega)$, where $\overline{p}$ is the constant vector $(p_1, ..., p_N)$. The theory of such spaces was developed by several authors, including [17, 18, 21].

We introduce $\overline{P}_+, \overline{P}_- \in R^N$ as

$$\overline{P}_+ = (p_1^+, ..., p_N^+),\quad \overline{P}_- = (p_1^-, ..., p_N^-),$$

and $P_+^+, P_-^+, P_-^- \in R^+$ as

$$P_+^+ = \max\{p_1^+, ..., p_N^+\},\quad P_-^+ = \max\{p_1^-, ..., p_N^-\},\quad P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

Throughout this paper we assume that $\sum_{i=1}^N \frac{1}{p_i^-} > 1$, and define $P_-^\star \in R^+$ and $P_{-,\infty} \in R^+$ by

$$P_-^\star = \frac{N}{\sum_{i=1}^N 1/p_i^- - 1},\qquad P_{-,\infty} = \max\{P_-^+, P_-^\star\}.$$

We recall that if $s \in C_+(\overline{\Omega})$ satisfies $1 < s(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,\overline{p}(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact (see [13,Theorem 1]).

Existence and multiplicity of solutions. By a *weak solution* to problem (1) we mean a function $u \in W_0^{1,\overline{p}(\cdot)}(\Omega)$ such that

$$\int_\Omega \left\{\sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}\phi - \lambda(x)|u|^{q(x)-2}u\phi\right\} dx = 0$$

for all $\phi \in W_0^{1,\overline{p}(\cdot)}(\Omega)$.

The following assumptions are required:

• $a_i(x,t)$ satisfy some growth and convexity conditions, so that they are of the type $|t|^{p_i(x)-2}t$ with $p_i \in C(\overline{\Omega})$, $p_i(x) \geq 2$ for all $x \in \overline{\Omega}$, $i = 1,...,N$.

• Function λ vanishes on a closed shell $\subset \Omega$ (for example, the region between two concentric spheres $B_r(x_0)$ and $B_R(x_0)$, $0 < r < R$), while λ is positive outside of the shell.

• $q \in C(\overline{\Omega})$ and $1 \leq q(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$.

• either $\max_{\overline{B_r(x_0)}} q < P_-^- \leq P_+^+ < \min_{\overline{\Omega\setminus B_R(x_0)}} q$, or, $\max_{\overline{\Omega\setminus B_R(x_0)}} q < P_-^- \leq P_+^+ < \min_{\overline{B_r(x_0)}} q$.

Theorem 1. *Under the above assumptions there exists a $\lambda^* > 0$ such that problem (1) has two positive nontrivial solutions for each λ with $|\lambda|_{L^\infty(\Omega)} < \lambda^*$.*

The eigenvalue problem. Here we assume that λ is a positive constant. Denote by w_i the width of Ω in the e_i direction, i.e., $w_i = \sup_{x,y\in\Omega}(x-y,e_i)$.

Our assumptions on a_i, q, p_i will be the following:

• $a_i(x,t) = |t|^{p_i(x)-2}t$ with $p_i \in C(\overline{\Omega})$, $p_i(x) \geq 2$ for all $x \in \overline{\Omega}$, $i = 1,...,N$.

• there exists a $j \in \{1,...,N\}$ such that $q(x) = q(x_1,...,x_{j-1},x_{j+1},...,x_N)$ (i.e. q is independent of x_j) and $p_j(x) = q(x)$ for all $x \in \overline{\Omega}$. Moreover, $q(x) < \frac{2}{w_j} + 1, \quad \forall\, x \in \overline{\Omega}$.

• there exists a $k \in \{1,...,N\}$ ($k \neq j$ above) such that $\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p_k(x)$.

Define

$$\lambda_1 = \inf_{u\in W_0^{1,\overline{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_i u|^{p_i(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx},$$

$$\lambda_0 = \inf_{u\in W_0^{1,\overline{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^N |\partial_i u|^{p_i(x)}\,dx}{\displaystyle\int_\Omega |u|^{q(x)}\,dx}.$$

Theorem 2. *In addition to the above conditions, assume $q(x) < P_-^\star$ for all $x \in \overline{\Omega}$. Then $0 < \lambda_0 \leq \lambda_1$ and every $\lambda \in (\lambda_1,\infty)$ is an eigenvalue of problem (1), while no $\lambda \in (0,\lambda_0)$ can be an eigenvalue of problem (1).*

REFERENCES

1. A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.* **14** (1973), 349–381.
2. Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.* **66** (2006), 1383-1406.
3. L. Diening, *Theoretical and Numerical Results for Electrorheological Fluids*, Ph.D. thesis, University of Frieburg, Germany, 2002.
4. D.E. Edmunds, J. Lang, and A. Nekvinda, On $L^{p(x)}$ norms, *Proc. Roy. Soc. London Ser. A* **455** (1999), 219–225.
5. D.E. Edmunds and J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. Roy. Soc. London Ser. A* **437** (1992), 229–236.
6. D.E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, *Studia Math.* **143** (2000), 267–293.
7. I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324–353.
8. I. Fragalà, F. Gazzola, and B. Kawohl, Existence and nonexistence results for anisotropic quasilinear equations, *Ann. Inst. H. Poincaré, Analyse Non Linéaire* **21** (2004), 715–734.
9. T. C. Halsey, Electrorheological fluids, *Science* **258** (1992), 761–766.
10. O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (1991), 592–618.
11. M. Mihăilescu and G. Moroşanu, Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions, in preparation.
12. M. Mihăilescu and G. Moroşanu, On an eigenvalue problem for an anisotropic elliptic problem involving variable exponents, in preparation.
13. M. Mihăilescu, P. Pucci and V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.* **340** (2008), 687-698.

14. M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A* **462** (2006), 2625–2641.
15. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. **1034**, Springer, Berlin, 1983.
16. K.R. Rajagopal and M. Ruzicka, Mathematical modelling of electrorheological fluids, *Continuum Mech. Thermodyn.* **13** (2001), 59–78.
17. J. Rákosník, Some remarks to anisotropic Sobolev spaces I, *Beiträge zur Analysis* **13** (1979), 55–68.
18. J. Rákosník, Some remarks to anisotropic Sobolev spaces II, *Beiträge zur Analysis* **15** (1981), 127–140.
19. M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer–Verlag, Berlin, 2002.
20. S. Samko and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, *J. Math. Anal. Appl.* **310** (2005), 229–246.
21. M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche Mat.* **18** (1969), 3–24.
22. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.* **29** (1987), 33–66.