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On a class of boundary value problems involving the p-biharmonic operator

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ABSTRACT

A nonlinear boundary value problem involving the p-biharmonic operator is investigated, where $p > 1$. It describes various problems in the theory of elasticity, e.g., the shape of an elastic beam where the bending moment depends on the curvature as a power function with exponent $p-1$. We prove existence of solutions satisfying a quite general boundary condition that incorporates many particular boundary conditions which are frequently considered in the literature.

1. Introduction

It is the purpose of this paper to investigate the following nonlinear, nonsmooth, fourth order boundary value problem

$$\left(\left|u''\right|^{p-2}u''\right)'' - \left(a(t)\left|u'\right|^{p-2}u'\right)' + b(t)|u|^{p-2}u \in \bar{\partial} F(t,u), \tag{1}$$

$$\begin{pmatrix} -(|u''|^{p-2}u'')'(0) + a(0)|u'(0)|^{p-2}u'(0) \\ (|u''|^{p-2}u'')'(1) - a(1)|u'(1)|^{p-2}u'(1) \\ |u''(0)|^{p-2}u''(0) \\ -|u''(1)|^{p-2}u''(1) \end{pmatrix} \in \partial j \begin{pmatrix} u(0) \\ u(1) \\ u'(0) \\ u'(1) \end{pmatrix}, \tag{2}$$

where $a, b \in C^0([0,1])$ are given real functions, $p > 1$ and F, j are nonlinear functions satisfying some conditions which are specified below. Both Eq. (1) and the boundary condition (2) are sufficiently general to cover a broad range of specific problems. Our treatment is mainly based on a variational approach.

To be more specific, let us formulate our assumptions on F and j:

(H_1) $F = F(t,\xi) : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, satisfying in addition $F(t,0) = 0$ for a.a. $t \in (0,1)$, as well as the Lipschitz condition:
$\forall \rho > 0$ there is an $\alpha_\rho \in L^1(0,1)$ such that

$$\left|F(t,x) - F(t,y)\right| \leqslant \alpha_\rho(t)|x-y|,$$

for a.a. $t \in (0,1)$ and all x, y with $|x|, |y| \leqslant \rho$;

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(H_2) Function $j:\mathbb{R}^4 \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous (l.s.c.), such that the null (column) vector $(0,0,0,0)^T \in D(j)$.

Now, to complete the presentation of our boundary value problem, let us explain the notation we have used above in (1) and (2). $\bar{\partial}F(t,\xi)$ denotes the generalized Clarke gradient of $F(t,\cdot)$ at $\xi \in \mathbb{R}$, while ∂j stands for the subdifferential of j (see [14]).

Note that our conditions $F(t,0)=0$ and $(0,0,0,0)^T \in D(j)$ do not restrict much the generality of the problem. In fact, the later can always be reached by a translation of u. Of course, this operation changes Eq. (1), but the treatment is similar. In particular, if either $p=2$ or b is the null function, Eq. (1) remains unchanged and it suffices to assume that $F(t,0)$ is an L^1 function.

A classical fourth order equation arising in the beam-column theory is the following (see Timoshenko and Gere [15])

$$EI\frac{d^4u}{dx^4} + P\frac{d^2u}{dx^2} = q, \tag{3}$$

where u is the lateral deflection, q is the intensity of a distributed lateral load, P is the axial compressive force applied to the beam and EI represents the flexural rigidity in the plane of bending. Eq. (3) is derived from the static equilibrium equations for any slice at distance x along the beam, namely the equilibrium of forces reads

$$q = -\frac{dV}{dx}, \tag{4}$$

where V is the shearing force and the equilibrium of moments is expressed by the equation

$$V = \frac{dM}{dx} - P\frac{du}{dx}, \tag{5}$$

where M denotes the bending moment. It is assumed that the bending moment depends linearly on the curvature. It can be expressed (if some higher order terms are neglected) as follows

$$EI\frac{d^2u}{dx^2} = -M. \tag{6}$$

Let us consider a more general situation, that the bending moment is a power function of the curvature with exponent $p-1$, i.e.,

$$M = -c\left|\frac{d^2u}{dx^2}\right|^{p-2}\frac{d^2u}{dx^2}, \tag{7}$$

where c is a constant. Then the presence of the term $(|u''|^{p-2}u'')''$ in (1) is justified if we assume (7) instead of (6) when Eq. (3) is derived. If $p=2$, then (7) coincides with (6) where $c=EI$.

Another equation that motivates our investigation here is the following one

$$Dw^{iv} + N_x w'' + Eh\frac{w}{a^2} = q, \quad t \in (0,1). \tag{8}$$

It models the radial deflection w for symmetrical buckling of a cylindrical shell under uniform axial compression N_x (see [15, p. 457], [13]).

The applied lateral load q in (3) or (8) represents the reaction of a support, which generally depends nonlinearly on the deflection (see [3–8,12,13]),

$$q(t) = f\big(t, u(t)\big),$$

or, more generally,

$$q(t) \in \bar{\partial}F\big(t, u(t)\big),$$

where F is a nonsmooth function (in particular, F may have some jumps, e.g., the case of adhesive support, see [13]).

Condition (2) covers many different types of boundary conditions (see [9]). For example, it is easy to check that for

$$j\big((x_1,x_2,x_3,x_4)^T\big) := \begin{cases} 0, & x_1 = x_2,\ x_3 = x_4, \\ +\infty, & \text{otherwise}, \end{cases}$$

we obtain the periodic conditions $u^{(i)}(0) = u^{(i)}(1)$, $i=0,1,2,3$, while the case of simply supported endpoints, i.e., $u(0) = u(1) = u''(0) = u''(1) = 0$, corresponds to the following choice

$$j\big((x_1,x_2,x_3,x_4)^T\big) := \begin{cases} 0, & x_1 = x_2 = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

We encourage the reader to find j for other types of classical boundary conditions (in particular for $p=2$ and a, b constants).

Our aim in this paper is to investigate the existence of solutions to problem (1)–(2). We extend our previous results related to the particular case $p = 2$ and a, b constants [6]. The treatment of the more general problem (1), (2) requires more advanced analysis.

By a *solution* of this problem we mean a function $u \in W^{2,p}(0,1)$, with $(|u''|^{p-2}u'')' \in \mathcal{AC}([0,1],\mathbb{R})$, which satisfies (2) and for a.a. $t \in (0,1)$

$$\big(\big|u''(t)\big|^{p-2}u''(t)\big)'' - \big(a(t)\big|u'(t)\big|^{p-2}u'(t)\big)' + b(t)\big|u(t)\big|^{p-2}u(t) \in \bar{\partial} F(t,u). \tag{9}$$

Here, $\mathcal{AC}([0,1],\mathbb{R})$ denotes the space of all absolutely continuous real functions defined on $[0,1]$. In fact, since $|u''|^{p-2}u'' =: v \in W^{2,1}(0,1)$, and $u'' = |v|^{q-2}v$, where q is the conjugate of p (i.e., $p^{-1} + q^{-1} = 1$), it follows that $u \in C^2([0,1])$. In particular, the values of u, u', u'' at $t = 0$ and $t = 1$ in (2) make sense. Note that if $1 < p \leqslant 2$ then $u \in C^3([0,1])$.

Now, we define the set

$$\mathcal{D} = \big\{u\colon\ u \in W^{2,p}(0,1),\ \big(u(0),u(1),u'(0),u'(1)\big)^T \in D(j)\big\},$$

and the functional $J : W^{2,p}(0,1) \to \mathbb{R} \cup \{+\infty\}$,

$$J(u) := j\big(\big(u(0),u(1),u'(0),u'(1)\big)^T\big), \quad \forall u \in W^{2,p}(0,1),$$

whose effective domain is $D(J) = \mathcal{D}$.

Obviously, $\mathcal{D} \neq \emptyset$ since $(0,0,0,0)^T \in D(j)$, so J is proper, convex and l.s.c.

In order to obtain existence of solutions to problem (1), (2), we consider the following functional

$$I(u) := \frac{1}{p}\int_0^1 \big(\big|u''\big|^p + a\big|u'\big|^p + b|u|^p\big)\,dt - \int_0^1 F(t,u)\,dt + J(u),$$

and use a technique similar to that developed in Motreanu and Panagiotopoulos [13].

Let us define the following two constants,

$$\lambda_1 := \inf\left\{\frac{\int_0^1 (|u''|^p + a|u'|^p + b|u|^p)\,dt}{\|u\|_{L^p}^p}\colon\ u \in \mathcal{D}\backslash\{0\}\right\}, \tag{10}$$

and

$$\bar{\lambda}_1 := \lim_{s\to\infty} \inf_{\substack{ru \in \mathcal{D}\\ r \geqslant s}} \left\{\int_0^1 \big(\big|u''\big|^p + a\big|u'\big|^p + b|u|^p\big)\,dt + \frac{pJ(ru)}{r^p}\colon\ \|u\|_{L^p}^p = 1\right\}. \tag{11}$$

Both λ_1 and $\bar{\lambda}_1$ will be important in the sequel. It is easily seen that $\lambda_1 \leqslant \bar{\lambda}_1$, but in most cases $\lambda_1 < \bar{\lambda}_1$.

We are now able to state the main results of the present paper, as follows.

Theorem 1.1. *Assume* (H_1) *and* (H_2). *Suppose, in addition, that the following condition is satisfied*

$$(L_\infty) \qquad \limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^p} < \frac{\lambda_1}{p}, \tag{12}$$

uniformly for a.a. $t \in (0,1)$. *Then problem* (1), (2) *has at least a solution.*

In order to state our next result, we introduce a new condition on F:

$$(L_0) \qquad \limsup_{x\to 0} \frac{F(t,x)}{|x|^p} < \frac{\lambda_1}{p},$$

uniformly for a.e. $t \in (0,1)$. Obviously this implies $0 \in \bar{\partial} F(t,0)$ for a.a. $t \in (0,1)$, so in this case $u(t) \equiv 0$ is a solution of problem (1), (2). We are interested in the existence of nontrivial solutions of problem (1), (2). We have

Theorem 1.2. *Assume that* $\lambda_1 > 0$ *and that* (L_0), (H_1), *and* (H_2) *are fulfilled. Suppose, in addition, that* $D(j)$ *is closed,* $(0,0,0,0)^T \in \partial j((0,0,0,0)^T)$, *and either* (G_θ) *or* (G_p)–$(\bar{L}_\infty)$ *holds, where*

(G_θ) *there exist constants* $\theta > p$, *and* $k, M > 0$, *such that*

$$j'(z;z) \leqslant \theta j(z) + k, \quad \forall z \in D(j), \tag{13}$$

$$0 < \theta F(t,x) \leqslant \xi x, \quad \forall \xi \in \bar{\partial} F(t,x), \tag{14}$$

for all $|x| > M$, *and a.a.* $t \in (0,1)$,

(G_p) *there exist positive constants* c, k, M *such that*

$$j'(z;z) \leqslant pj(z) + k, \quad \forall z \in D(j), \tag{15}$$

$$0 < \left(p + \frac{c}{|x|^{p-1}}\right) F(t,x) \leqslant \xi x, \quad \forall \xi \in \bar{\partial} F(t,x), \tag{16}$$

for all $|x| > M$, *and a.a.* $t \in (0,1)$, *and*

$$(\bar{L}_\infty) \qquad \liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p} > \frac{\bar{\lambda}_1}{p}, \tag{17}$$

uniformly for a.a. $t \in (0,1)$.

Then problem (1), (2) *has at least a nonzero solution.*

Remark 1.1. It is worth pointing out that any of the conditions (13) and (15) implies that the domain $D(j)$ of functional j is a convex cone. Moreover, assumption (15) guarantees that $\bar{\lambda}_1 < \infty$ (see Lemma 2.2 below).

2. Preliminaries

Let X be a Banach space whose dual is denoted by X^*. We recall that the generalized directional derivative $\Phi^0(u;v)$ of a locally Lipschitz function $\Phi : X \to \mathbb{R}$ at a point $u \in X$ and in the direction $v \in X$ is defined by

$$\Phi^0(u;v) := \limsup_{w\to u,\, s\downarrow 0} \frac{\Phi(w+sv) - \Phi(w)}{s}.$$

The set

$$\bar{\partial}\Phi(u) := \left\{\eta \in X^*\colon\ \Phi^0(u;v) \geqslant \langle \eta, v\rangle,\ \forall v \in X\right\}$$

denotes the generalized gradient $\bar{\partial}\Phi(u)$ of the function Φ (in the sense of Clarke [1,2]).

Assume that the functional I has the form

$$I = \Phi + \psi,$$

where $\Phi : X \to \mathbb{R}$ is a locally Lipschitz function, and $\psi : X \to (-\infty, +\infty]$ is a proper, convex and l.s.c. function. We recall the definition of a critical point of the functional I as well as the Palais–Smale condition.

Definition 2.1. A vector $u \in X$ is said to be a critical point of the functional $I = \Phi + \psi$ if the following inequality holds

$$\Phi^0(u; v-u) + \psi(v) - \psi(u) \geqslant 0, \quad \forall v \in X.$$

A number $c \in \mathbb{R}$ such that $I^{-1}(c)$ contains a critical point is called a critical value of I.

Definition 2.2. The functional I is said to satisfy the Palais–Smale (*PS*) condition if every sequence $\{u_n\} \subset X$ such that $|I(u_n)| < C$ with a constant C and

$$\Phi^0(u_n; v-u_n) + \psi(v) - \psi(u_n) \geqslant -\varepsilon_n \|v-u_n\|, \quad \forall v \in X, \tag{18}$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$, possesses a convergent subsequence.

In what follows we need the following generalized mountain pass theorem (cf. [13]; see also [11] and [16]).

Theorem 2.1 *(Mountain pass). Suppose that I satisfies the* (*PS*) *condition,* $I(0) = 0$ *and*

(i) *there exist* $\alpha, \rho > 0$ *such that* $I(u) \geqslant \alpha$ *if* $\|u\| = \rho$,
(ii) $I(e) \leqslant 0$ *for some* $e \in X$, *with* $\|e\| > \rho$.

Then the number

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I\big(\gamma(t)\big),$$

where

$$\Gamma = \left\{\gamma \in C\big([0,1], X\big)\colon\ \gamma(0) = 0,\ \gamma(1) = e\right\}$$

is a critical value of I with $c \geqslant \alpha$.

Our framework in the next part of the paper will be $X = W^{2,p}(0,1)$.

Define $\psi : X \to (-\infty, +\infty]$ by

$$\psi(u) := \frac{1}{p}\int_0^1 \left(\left|u''\right|^p + \left|u'\right|^p + |u|^p\right) dt + J(u) = \frac{1}{p}\|u\|^p_{W^{2,p}(0,1)} + J(u),$$

and

$$\varphi(u) := \frac{1}{p}\int_0^1 \left(\left(a(t)-1\right)\left|u'(t)\right|^p + \left(b(t)-1\right)\left|u(t)\right|^p\right) dt, \quad u \in X.$$

Notice that ψ is a proper, convex and l.s.c. functional whose effective domain is $D(\psi) = \mathcal{D}$, while $\varphi \in C^1(W^{2,p}(0,1), \mathbb{R})$, and

$$\langle \varphi'(u), v \rangle = \int_0^1 \left(\left(a(t)-1\right)\left|u'(t)\right|^{p-2} u'(t)v'(t) + \left(b(t)-1\right)\left|u(t)\right|^{p-2} u(t)v(t)\right) dt.$$

The following proposition characterizes the generalized gradient $\bar{\partial}\Phi(u)$ of the functional

$$\Phi(u) := -\int_0^1 F(t,u)\,dt + \varphi(u), \quad u \in X = W^{2,p}(0,1). \tag{19}$$

Proposition 2.1. *Assume that $F : (0,1) \times \mathbb{R} \to \mathbb{R}$ satisfies (H_1). Then the functional Φ defined by (19) is locally Lipschitz. Moreover, if $u \in W^{2,p}(0,1)$ and $l \in \bar{\partial}\Phi(u)$ then there is some $u_l \in L^1(0,1)$ such that $u_l(t) \in \bar{\partial}F(t,u(t))$ for a.a. $t \in (0,1)$, and*

$$\langle l, v \rangle = \int_0^1 \left(-u_l(t)v(t) + \left(a(t)-1\right)\left|u'(t)\right|^{p-2} u'(t)v'(t) + \left(b(t)-1\right)\left|u(t)\right|^{p-2} u(t)v(t)\right) dt, \quad \forall v \in W^{2,p}(0,1), \tag{20}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing in $W^{2,p}(0,1)$.

Proof. By the continuity of the embedding $W^{2,p}(0,1) \subset C^1([0,1])$ and assumption (H_1) it follows that Φ is indeed locally Lipschitz.

Define

$$R(u) := -\int_0^1 F\left(t,u(t)\right) dt, \quad \forall u \in W^{2,p}(0,1).$$

One can prove (see Theorem 2.7.3 in [2]) that given $\xi \in \bar{\partial}R(u)$ then there exists some $v_\xi \in L^1(0,1)$ such that $v_\xi(t) \in -\bar{\partial}F(t,u(t))$ for a.a. $t \in (0,1)$, and

$$\langle \xi, v \rangle = \int_0^1 v_\xi v\,dt, \quad \forall v \in W^{2,p}(0,1). \tag{21}$$

For a complete direct proof of this assertion we refer the reader to [6, p. 2804].

Now, let $l \in \bar{\partial}\Phi(u)$. By $\bar{\partial}\Phi(u) \subset \bar{\partial}R(u) + \bar{\partial}\varphi(u) = \bar{\partial}R(u) + \varphi'(u)$, there exists $\xi \in \bar{\partial}R(u)$ such that $l = \xi + \varphi'(u)$ and (20) is obtained with $u_l := -v_\xi$, where v_ξ is determined by ξ as above. □

Define $I : X = W^{2,p}(0,1) \to (-\infty, +\infty]$ by

$$I(u) := \Phi(u) + \psi(u) = \frac{1}{p}\int_0^1 \left(\left|u''\right|^p + a\left|u'\right|^p + b|u|^p\right) dt - \int_0^1 F(t,u)\,dt + J(u).$$

Theorem 2.2. *If $F:(0,1)\times\mathbb{R}\to\mathbb{R}$ satisfies (H_1) and $u\in W^{2,p}(0,1)$ is a critical point of functional I, then u is a solution of problem* (1), (2).

Proof. We adapt a previous device from [10, Proposition 3.2], to the present functional (see also [6, p. 2805]). If we take $v=u+sw$, $s>0$, in the inequality

$$\Phi^0(u;v-u)+\psi(v)-\psi(u)\geqslant 0,\quad \forall v\in W^{2,p}(0,1),$$

we easily get

$$\Phi^0(u;w)+\int_0^1\big(\big|u''\big|^{p-2}u''w''+\big|u'\big|^{p-2}u'w'+|u|^{p-2}uw\big)\,dt+J'(u;w)\geqslant 0, \tag{22}$$

for all $w\in W^{2,p}(0,1)$, where

$$J'(u;w)=j'\big(\big(u(0),u(1),u'(0),u'(1)\big)^T;\big(w(0),w(1),w'(0),w'(1)\big)^T\big) \tag{23}$$

is the directional derivative of the convex function J at u in the direction w. For $w\in C_0^\infty(0,1)\subset W^{2,p}(0,1)$, inequality (22) reads

$$\Phi^0(u;w)\geqslant-\int_0^1\big(\big|u''\big|^{p-2}u''w''+\big|u'\big|^{p-2}u'w'+|u|^{p-2}uw\big)\,dt. \tag{24}$$

The Hahn–Banach theorem implies that there exists a linear functional $l:W^{2,p}(0,1)\to\mathbb{R}$, such that

$$\Phi^0(u;w)\geqslant l(w),\quad \forall w\in W^{2,p}(0,1), \tag{25}$$

and

$$l(w)=-\int_0^1\big(\big|u''\big|^{p-2}u''w''+\big|u'\big|^{p-2}u'w'+|u|^{p-2}uw\big)\,dt,\quad \forall w\in C_0^\infty(0,1)$$

as far as the function $\Phi^0(u;\cdot)$ is subadditive and positively homogeneous. Moreover, the estimate

$$\Phi^0(u;w)\leqslant k\|w\|_{W^{2,p}(0,1)},\quad \forall w\in W^{2,p}(0,1) \tag{26}$$

holds with $k>0$ being a Lipschitz constant of Φ in a vicinity of u. Hence,

$$\big|l(w)\big|\leqslant k\|w\|_{W^{2,p}(0,1)},\quad \forall w\in W^{2,p}(0,1),$$

showing that l is continuous. The inequality (25) yields that $l\in\bar\partial\Phi(u)$. Now, there is some $u_l\in L^1(0,1)$ such that

$$u_l(t)\in\bar\partial F\big(t,u(t)\big),\quad \text{for a.a. } t\in(0,1), \tag{27}$$

and

$$\int_0^1\big(\big|u''\big|^{p-2}u''w''+a\big|u'\big|^{p-2}u'w'+b|u|^{p-2}uw-u_lw\big)\,dt=0, \tag{28}$$

for all $w\in C_0^\infty(0,1)$ as a consequence of Proposition 2.1. Since $u\in W^{2,p}(0,1)$, we have $(|u''|^{p-2}u'')'\in W^{1,1}(0,1)$, i.e., $(|u''|^{p-2}u'')'$ is absolutely continuous and

$$\big(\big|u''(t)\big|^{p-2}u''(t)\big)''-\big(a(t)\big|u'(t)\big|^{p-2}u'(t)\big)'+b(t)\big|u(t)\big|^{p-2}u(t)=u_l(t) \tag{29}$$

for a.a. $t\in(0,1)$. Then (9) easily follows from (27).

Next, we prove that u satisfies (2). We already know that $u''\in C^2([0,1])$. The above inclusion relation (27) implies that

$$u_l(t)w(t)\leqslant F^0\big(t,u(t);w(t)\big)\quad \text{for a.a. } t\in(0,1),\ \forall w\in W^{2,p}(0,1).$$

Then, by (29), we have

$$\int_0^1 \big(|u''|^{p-2}u''w'' + a|u'|^{p-2}u'w' + b|u|^{p-2}uw\big)\,dt + \big(\big(|u''|^{p-2}u''\big)'(1) - a(1)|u'(1)|^{p-2}u'(1)\big)w(1)$$
$$- \big(\big(|u''|^{p-2}u''\big)'(0) - a(0)|u'(0)|^{p-2}u'(0)\big)w(0) - |u''(1)|^{p-2}u''(1)w'(1) + |u''(0)|^{p-2}u''(0)w'(0)$$
$$\leqslant \int_0^1 F^0\big(t, u(t); w(t)\big)\,dt,$$

for all $w \in W^{2,p}(0,1)$. Thus,

$$\Phi^0(u; w) \leqslant \int_0^1 (-F)^0\big(t, u(t); w(t)\big)\,dt + \langle \varphi'(u), w\rangle,$$

and from (22), we get

$$\int_0^1 (-F)^0\big(t, u(t); w(t)\big)\,dt - \int_0^1 F^0\big(t, u(t); w(t)\big)\,dt + J'(u; w)$$
$$\geqslant \big(\big(|u''|^{p-2}u''\big)'(1) - a(1)|u'(1)|^{p-2}u'(1)\big)w(1) - \big(\big(|u''|^{p-2}u''\big)'(0) - a(0)|u'(0)|^{p-2}u'(0)\big)w(0)$$
$$- |u''(1)|^{p-2}u''(1)w'(1) + |u''(0)|^{p-2}u''(0)w'(0), \tag{30}$$

for all $w \in W^{2,p}(0,1)$. Assume that $x, y, z, q \in \mathbb{R}$ are arbitrary constants. Let $w_n \in W^{2,p}(0,1)$, $n \in \mathbb{N}$, be defined by

$$w_n := \begin{cases} x\omega_0(nt) + \frac{y}{n}\omega_1(nt), & \text{if } t \in [0, \frac{1}{n}), \\ 0, & \text{if } t \in [\frac{1}{n}, 1 - \frac{1}{n}], \\ z\omega_0(n(1-t)) - \frac{q}{n}\omega_1(n(1-t)), & \text{if } t \in (1 - \frac{1}{n}, 1], \end{cases}$$

where $\omega_0(s)$ and $\omega_1(s)$ are such that $\omega_0(1) = \omega_1(1) = \omega_0'(1) = \omega_1'(1) = 0$, $\omega_0(0) = \omega_1'(0) = 1$ and $\omega_0'(0) = \omega_1(0) = 0$, e.g., $\omega_0(s) := (s-1)^2(2s+1)$ and $\omega_1(s) := s(s-1)^2$. It is easy to check that $w_n(0) = x$, $w_n'(0) = y$, $w_n(1) = z$ and $w_n'(1) = q$.

Since hypothesis (H_1) holds, there is $\alpha_\rho \in L^1(0,1)$ such that

$$\big|F^0\big(t, u(t); \eta\big)\big| \leqslant \alpha_\rho(t)|\eta|, \quad \forall \eta \in \mathbb{R}, \text{ for a.a. } t \in (0,1),$$

where $\rho > 0$ depends on the supremum norm $\|u\|_\infty$ of u. Thus, if in particular $\eta = w_n(t)$, one obtains

$$\big|F^0\big(t, u(t); w_n(t)\big)\big| \leqslant \alpha_\rho(t) \max\left\{|x| + \frac{4|y|}{27n}, |z| + \frac{4|q|}{27n}\right\}, \tag{31}$$

for a.a. $t \in (0,1)$. Then, by Lebesgue's dominated convergence theorem,

$$\int_0^1 F^0\big(t, u(t); w_n(t)\big)\,dt \to 0, \quad \text{as } n \to \infty, \tag{32}$$

in view of the fact that

$$F^0\big(t, u(t); w_n(t)\big) \to F^0\big(t, u(t); 0\big) = 0, \quad \text{for a.a. } t \in (0,1).$$

Similarly, one has

$$\int_0^1 (-F)^0\big(t, u(t); w_n(t)\big)\,dt \to 0, \quad \text{as } n \to \infty. \tag{33}$$

Finally, we take $w = w_n$ in (30) and let $n \to \infty$. Thus, by (23), (32) and (33), we derive

$$j'\big(\big(u(0), u(1), u'(0), u'(1)\big)^T; (x, z, y, q)^T\big) \geqslant \big(\big(|u''|^{p-2}u''\big)'(1) - a(1)|u'(1)|^{p-2}u'(1)\big)z$$
$$- \big(\big(|u''|^{p-2}u''\big)'(0) - a(0)|u'(0)|^{p-2}u'(0)\big)x$$
$$- |u''(1)|^{p-2}u''(1)q + |u''(0)|^{p-2}u''(0)y.$$

Since x, y, z, and q were arbitrarily chosen, it follows that u satisfies (2). □

Lemma 2.1. *We have $\lambda_1 > -\infty$, where λ_1 is the constant defined by* (10). *Moreover, if $\lambda_1 > 0$, then there exists a constant $m > 0$ such that*

$$\int_0^1 \left(\left|u''\right|^p + a\left|u'\right|^p + b|u|^p\right) dt \geqslant m\|u\|^p_{W^{2,p}(0,1)}, \quad \forall u \in \mathcal{D}.$$

Proof. First, there exists a constant K such that

$$\left\|u'\right\|^p_{L^p} \leqslant K\left(\varepsilon\left\|u''\right\|^p_{L^p} + \varepsilon^{-1}\|u\|^p_{L^p}\right), \quad \forall \varepsilon \in (0,1). \tag{34}$$

Choose ε such that $K\varepsilon \leqslant |a|^{-1}_\infty$ where $|a|_\infty := \max_{t\in[0,1]} |a(t)|$. Let λ be such that $b(t) + \lambda \geqslant |a|_\infty K\varepsilon^{-1}$. Then,

$$\left\|u'\right\|^p_{L^p} \leqslant K\left(\varepsilon\left\|u''\right\|^p_{L^p} + \varepsilon^{-1}\|u\|^p_{L^p}\right) \leqslant |a|^{-1}_\infty \int_0^1 \left(\left|u''\right|^p + (b+\lambda)|u|^p\right) dt,$$

i.e.,

$$\int_0^1 \left(\left|u''\right|^p + a\left|u'\right|^p + b|u|^p\right) dt \geqslant \int_0^1 \left(\left|u''\right|^p - |a|_\infty\left|u'\right|^p + b|u|^p\right) dt \geqslant -\lambda\|u\|^p_{L^p},$$

so $\lambda_1 \geqslant -\lambda > -\infty$.

Next, suppose that $\lambda_1 > 0$ holds. We will prove that there is some $\mu > 0$ such that

$$\int_0^1 \left(\left|u''\right|^p + a\left|u'\right|^p + b|u|^p\right) dt \geqslant \frac{\mu}{\mu+1}\left(\left\|u''\right\|^p_{L^p} + \left\|u'\right\|^p_{L^p} + \|u\|^p_{L^p}\right), \tag{35}$$

for all $u \in \mathcal{D}$. The inequality (35) is equivalent to

$$\int_0^1 \left(\left|u''\right|^p + \left((a-1)\mu + a\right)\left|u'\right|^p + \left((b-1)\mu + b\right)|u|^p\right) dt \geqslant 0.$$

We will prove that there is $\mu > 0$ such that the following stronger inequality

$$\int_0^1 \left(\left|u''\right|^p + \left(-\left(|a|_\infty + 1\right)\mu + a\right)\left|u'\right|^p + \left(-\left(|b|_\infty + 1\right)\mu + b\right)|u|^p\right) dt \geqslant 0$$

holds for every $u \in \mathcal{D}$. Let K be as above and ε, $0 < \varepsilon < 1$ be such that $K\varepsilon|a|_\infty < 1$. We denote with δ, $0 < \delta < 1$ a number that will be chosen later. We have

$$\begin{aligned}
&\int_0^1 \left(\left|u''\right|^p + \left(-\left(|a|_\infty + 1\right)\mu + a\right)\left|u'\right|^p + \left(-\left(|b|_\infty + 1\right)\mu + b\right)|u|^p\right) dt \\
&\quad \geqslant (1-\delta)\int_0^1 \left(\left|u''\right|^p + a\left|u'\right|^p + b|u|^p\right) dt - \left(|b|_\infty\delta + \left(|b|_\infty + 1\right)\mu\right)\int_0^1 |u|^p\, dt \\
&\qquad + \delta\int_0^1 \left|u''\right|^p dt - \left(|a|_\infty\delta + \left(|a|_\infty + 1\right)\mu\right)\int_0^1 \left|u'\right|^p dt \\
&\quad \geqslant \left((1-\delta)\lambda_1 - \left(|b|_\infty\delta + \left(|b|_\infty + 1\right)\mu\right)\right)\int_0^1 |u|^p\, dt + \delta\int_0^1 \left|u''\right|^p dt - \left(|a|_\infty\delta + \left(|a|_\infty + 1\right)\mu\right)\int_0^1 \left|u'\right|^p dt
\end{aligned}$$

for every $u \in \mathcal{D}$. We look for μ and δ such that the inequalities

$$\delta \geqslant \left(|a|_\infty\delta + \left(|a|_\infty + 1\right)\mu\right)K\varepsilon, \tag{36}$$

$$A := (1-\delta)\lambda_1 - \left(|b|_\infty\delta + \left(|b|_\infty + 1\right)\mu\right) \geqslant \left(|a|_\infty\delta + \left(|a|_\infty + 1\right)\mu\right)K\varepsilon^{-1} \tag{37}$$

are satisfied. The first inequality is equivalent to

$$\frac{1-K\varepsilon|a|_\infty}{(|a|_\infty+1)K\varepsilon}\delta \geqslant \mu \tag{38}$$

and the second one is equivalent to

$$\lambda_1 \geqslant \big(\lambda_1+|b|_\infty+|a|_\infty K\varepsilon^{-1}\big)\delta+\big(1+|b|_\infty+\big(|a|_\infty+1\big)K\varepsilon^{-1}\big)\mu. \tag{39}$$

Obviously there are μ and δ such that (38) and (39) hold as well as the inequalities (36) and (37). Therefore,

$$\begin{aligned}&\int_0^1 \big(\big|u''\big|^p+\big(-\big(|a|_\infty+1\big)\mu+a\big)\big|u'\big|^p+\big(-\big(|b|_\infty+1\big)\mu+b\big)|u|^p\big)\,dt\\ &\quad\geqslant A\int_0^1|u|^p\,dt+\delta\int_0^1\big|u''\big|^p\,dt-\big(|a|_\infty\delta+\big(|a|_\infty+1\big)\mu\big)\int_0^1\big|u'\big|^p\,dt\\ &\quad\geqslant A\int_0^1|u|^p\,dt+\delta\int_0^1\big|u''\big|^p\,dt-\big(|a|_\infty\delta+\big(|a|_\infty+1\big)\mu\big)\left(K\varepsilon\int_0^1\big|u''\big|^p\,dt+K\varepsilon^{-1}\int_0^1|u|^p\,dt\right)\geqslant 0. \qquad\square\end{aligned}$$

Lemma 2.2. *Assume that* (15) *holds. Then* $\bar\lambda_1<+\infty$.

Proof. The inequality (15) implies that

$$s^{-p}j(sz)\leqslant j(z)+\frac{k}{p}\big(1-s^{-p}\big),\quad \forall z\in D(j),\ \forall s\geqslant 1.$$

Then,

$$\begin{aligned}\bar\lambda_1 \leqslant \inf\big\{&\big\|u''\big\|_{L^p}^p+|a|_\infty\big\|u'\big\|_{L^p}^p+|b|_\infty\|u\|_{L^p}^p+pJ(u)\colon u\in W^{2,p}(0,1),\ \|u\|_{L^p}^p=1,\\ &\big(u(0),u(1),u'(0),u'(1)\big)^T\in D(j)\big\}+k. \qquad\square\end{aligned}$$

Proof of Theorem 1.1. There are constants $\sigma>0$ and $\rho>0$ such that

$$F(t,x)\leqslant\frac{\lambda_1-\sigma}{p}|x|^p \quad\text{for a.a. } t\in(0,1),$$

and for all x with $|x|>\rho$, since inequality (12) holds. Hence,

$$F(t,x)\leqslant\rho\alpha_\rho(t)+\left|\frac{\lambda_1-\sigma}{p}\right|\rho^p+\frac{\lambda_1-\sigma}{p}|x|^p\quad\text{for a.a. } t\in(0,1),\ x\in\mathbb{R},$$

which gives

$$\Phi(u)-\varphi(u)=-\int_0^1F(t,u)\,dt\geqslant -k-\frac{\lambda_1-\sigma}{p}\int_0^1|u|^p\,dt,\quad \forall u\in W^{2,p}(0,1). \tag{40}$$

On the other hand, J is proper, convex and l.s.c. Therefore, it is bounded from below by an affine functional, i.e.,

$$J(u)\geqslant -c_1-c_2\|u\|, \tag{41}$$

with some constants $c_1>0$ and $c_2>0$. We use (40) and (41) to prove that functional I is coercive.

Suppose on the contrary that $\{u_n\}\subset W^{2,p}(0,1)$ is a sequence such that $\|u_n\|_{W^{2,p}(0,1)}\to\infty$ and $I(u_n)\leqslant C$ with a constant $C>0$. Denote $v_n:=\frac{u_n}{\|u_n\|}$, where $\|\cdot\|$ is the norm in $W^{2,p}(0,1)$. Then, $u_n, v_n\in D(J)$. We have

$$\begin{aligned}C&\geqslant\frac{1}{p}\int_0^1\big(\big|u_n''\big|^p+a\big|u_n'\big|^p+b|u_n|^p\big)\,dt+J(u_n)+\Phi(u_n)-\varphi(u_n)\\ &\geqslant\frac{1}{p}\int_0^1\big(\big|u_n''\big|^p+a\big|u_n'\big|^p+b|u_n|^p\big)\,dt+\Phi(u_n)-\varphi(u_n)-c_1-c_2\|u_n\|,\end{aligned}$$

which implies

$$\frac{C+c_1}{\|u_n\|^p}+\frac{c_2}{\|u_n\|^{p-1}}\geqslant\frac{1}{p}\int_0^1\left(\left|v_n''\right|^p+a\left|v_n'\right|^p+b|v_n|^p\right)dt+\frac{\Phi(u_n)-\varphi(u_n)}{\|u_n\|^p}. \tag{42}$$

Now, it follows

$$\begin{aligned}\frac{C+c_1+k}{\|u_n\|^p}+\frac{c_2}{\|u_n\|^{p-1}}&\geqslant\frac{1}{p}\int_0^1\left(\left|v_n''\right|^p+a\left|v_n'\right|^p+(b-\lambda_1+\sigma)|v_n|^p\right)dt\\&\geqslant\frac{\sigma}{p}\int_0^1|v_n|^p\,dt.\end{aligned} \tag{43}$$

Since $\|v_n\|=1$, there exists a subsequence of $\{v_n\}$ (denoted again by $\{v_n\}$) and $v\in D(J)$, such that $v_n\rightharpoonup v$ in $W^{2,p}(0,1)$. Therefore, $v_n\to v$ strongly in $C^1([0,1])$. Taking into account estimate (43), we deduce that in fact $v_n\to v=0$ in $C^1([0,1])$. Then,

$$\left\|v_n''\right\|_{L^p}^p=\|v_n\|_{W^{2,p}}^p-\left\|v_n'\right\|_{L^p}^p-\|v_n\|_{L^p}^p\to 1,\quad\text{as } n\to\infty.$$

This implies that

$$\frac{C+c_1+k}{\|u_n\|^p}+\frac{c_2}{\|u_n\|^{p-1}}\geqslant\frac{1}{p}\int_0^1\left(\left|v_n''\right|^p+a\left|v_n'\right|^p+(b-\lambda_1+\sigma)|v_n|^p\right)dt\to\frac{1}{p}$$

a contradiction. The coercivity of functional I is verified.

Next, the functional Φ is sequentially weakly continuous due to the compactness of the imbedding $W^{2,p}(0,1)\subset C^1([0,1])$. Hence, by the convexity of ψ, functional I is sequentially weakly lower semi-continuous and its coercivity implies that it is bounded from below and attains its infimum. Then I has a critical point, which by Theorem 2.2 is a solution of problem (1)–(2). □

Lemma 2.3. *Assume (H_1) holds, $\lambda_1>0$, and either (G_θ) or (G_p) holds. If, in addition, $D(j)$ is closed, then functional I satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\}$ be an arbitrary Palais–Smale sequence. Set $v=(1+s)u_n$ in the inequality (18), where $s>0$. Taking the limit as $s\to 0^+$, we obtain

$$\Phi^0(u_n;u_n)+\psi'(u_n;u_n)\geqslant-\varepsilon_n\|u_n\|.$$

This inequality reads

$$\Phi^0(u_n;u_n)-\left\langle\varphi'(u_n),u_n\right\rangle+\int_0^1\left(\left|u_n''\right|^p+a\left|u_n'\right|^p+b|u_n|^p\right)dt+J'(u_n;u_n)\geqslant-\varepsilon_n\|u_n\|. \tag{44}$$

Next, there exists a constant C such that

$$C\geqslant I(u_n)=\frac{1}{p}\int_0^1\left(\left|u_n''\right|^p+a\left|u_n'\right|^p+b|u_n|^p\right)dt+J(u_n)+\Phi(u_n)-\varphi(u_n). \tag{45}$$

We use (44) and (45) to prove that in fact the sequence $\{u_n\}$ is bounded. We will examine separately the cases when (G_p) and (G_θ) hold.

Case 1. Let (G_θ) hold for some $\theta>p$. First, we verify that

$$\theta\big(\Phi(u)-\varphi(u)\big)\geqslant\Phi^0(u;u)-\left\langle\varphi'(u),u\right\rangle-m_1,\quad\forall u\in W^{2,p}(0,1), \tag{46}$$

and

$$\theta J(u)\geqslant J'(u;u)-m_2,\quad\forall u\in W^{2,p}(0,1), \tag{47}$$

for some positive constants m_1 and m_2. The inequality (47) follows from the definition of the functional J and condition (13). Now, let $l\in\bar\partial\Phi(u)$. Then, there exists $u_l\in L^1$ (see Proposition 2.1), such that $u_l(t)\in\bar\partial F(t,u(t))$ for a.a. $t\in(0,1)$, and

$$\langle l, v\rangle = \int_0^1 \bigl(-u_l(t)v(t) + \bigl(a(t)-1\bigr)\bigl|u'(t)\bigr|^{p-2}u'(t)v'(t) + \bigl(b(t)-1\bigr)\bigl|u(t)\bigr|^{p-2}u(t)v(t)\bigr)\,dt, \quad \forall v \in W^{2,p}(0,1).$$

Next, hypothesis (H_1) says that given $M > 0$ there exists an $\alpha_M(t) \in L^1$ such that for each $x \in \mathbb{R}$, with $|x| \leqslant M$, the inequalities

$$|\xi| \leqslant \alpha_M(t), \quad \forall \xi \in \bar{\partial} F(t,x),$$

and

$$\bigl|F(t,x)\bigr| \leqslant M\alpha_M(t),$$

are satisfied. Hence,

$$\begin{aligned}\int_0^1 u(t)u_l(t)\,dt &= \int_{|u(t)|>M} u(t)u_l(t)\,dt + \int_{|u(t)|\leqslant M} u(t)u_l(t)\,dt \\ &\geqslant \int_{|u(t)|>M} \theta F\bigl(t,u(t)\bigr)\,dt - M\int_0^1 \alpha_M(t)\,dt \\ &= \theta\left(\int_0^1 F\bigl(t,u(t)\bigr)\,dt - \int_{|u(t)|\leqslant M} F\bigl(t,u(t)\bigr)\,dt\right) - M\int_0^1 \alpha_M(t)\,dt \\ &\geqslant \theta\int_0^1 F\bigl(t,u(t)\bigr)\,dt - M(1+\theta)\int_0^1 \alpha_M(t)\,dt.\end{aligned}$$

Obviously,

$$\int_0^1 F\bigl(t,u(t)\bigr)\,dt = -\Phi(u) + \varphi(u) \quad \text{and} \quad \int_0^1 u(t)u_l(t)\,dt = -\langle l, u\rangle + \bigl\langle \varphi'(u), u\bigr\rangle$$

and we get

$$\theta\bigl(\Phi(u) - \varphi(u)\bigr) \geqslant \langle l, u\rangle - \bigl\langle \varphi'(u), u\bigr\rangle - m_1, \quad \forall l \in \bar{\partial}\Phi(u),$$

where $m_1 = M(1+\theta)\int_0^1 \alpha_M(t)\,dt > 0$. Finally, it follows that

$$\begin{aligned}\theta\bigl(\Phi(u) - \varphi(u)\bigr) &\geqslant \max\bigl\{\langle l, v\rangle \colon l \in \bar{\partial}\Phi(u)\bigr\} - \bigl\langle \varphi'(u), u\bigr\rangle - m_1 \\ &= \Phi^0(u; u) - \bigl\langle \varphi'(u), u\bigr\rangle - m_1,\end{aligned}$$

which yields (46).

Now, setting $u = u_n$ in (46) and (47) and multiplying (45) by θ, one can derive from (44)–(47) that

$$\theta C + m_1 + m_2 \geqslant \left(\frac{\theta}{p} - 1\right)\int_0^1 \bigl(\bigl|u_n''\bigr|^p + a\bigl|u_n'\bigr|^p + b|u_n|^p\bigr)\,dt - \varepsilon_n\|u_n\|.$$

Finally, by the hypothesis $\lambda_1 > 0$ and by Lemma 2.1, there exists a constant $m_3 > 0$ such that

$$\theta C + m_1 + m_2 \geqslant m_3\|u_n\|^p - \varepsilon_n\|u_n\|,$$

which implies that $\{u_n\}$ is bounded.

Case 2. Let (G_p) hold. First, using hypothesis (16) we derive a similar to (46) inequality. More precisely, we show that there exists a constant $k_1 > 0$, and, given $\rho > 0$ there exists a constant $m_1 = m_1(\rho) > 0$, such that for each $u \in W^{2,p}(0,1)$, with $\|u\| \geqslant \rho$, the following

$$\left(p + \frac{k_1}{\|u\|^{p-1}}\right)\bigl(\Phi(u) - \varphi(u)\bigr) \geqslant \Phi^0(u; u) - \bigl\langle \varphi'(u), u\bigr\rangle - m_1 \tag{48}$$

holds. Let $l \in \bar{\partial}\Phi(u)$ and $u_l \in L^1$ be defined as in Case 1. Then,

$$\int_0^1 u(t)u_l(t)\,dt = \int_{\{|u(t)|>M\}} u(t)u_l(t)\,dt + \int_{\{|u(t)|\leqslant M\}} u(t)u_l(t)\,dt$$
$$\geqslant \int_{\{|u(t)|>M\}} \left(p + \frac{c}{|u(t)|^{p-1}}\right) F\big(t,u(t)\big) - M\int_0^1 \alpha_M(t)\,dt$$
$$\geqslant \left(p + \frac{c}{d\|u\|^{p-1}}\right) \int_{\{|u(t)|>M\}} F\big(t,u(t)\big) - M\int_0^1 \alpha_M(t)\,dt$$
$$= \left(p + \frac{c}{d\|u\|^{p-1}}\right)\left(\int_0^1 F\big(t,u(t)\big) - \int_{\{|u(t)|\leqslant M\}} F\big(t,u(t)\big)\right) - M\int_0^1 \alpha_M(t)\,dt$$
$$\geqslant \left(p + \frac{c}{d\|u\|^{p-1}}\right)\int_0^1 F\big(t,u(t)\big) - M\left(p+1+\frac{c}{d\rho^{p-1}}\right)\int_0^1 \alpha_M(t)\,dt,$$

where the positive constant d is such that $|u(t)| \leqslant d\|u\|_{W^{2,p}(0,1)}$. Similarly as in Case 1, we obtain that (48) holds with $k_1 = c/d$, and

$$m_1 = M\left(p+1+\frac{c}{d\rho^{p-1}}\right)\int_0^1 \alpha_M(t)\,dt.$$

Next, inequalities (15) and (41) imply

$$\left(p + \frac{k_1}{\|u\|^{p-1}}\right) J(u) \geqslant J'(u;u) - k + \frac{k_1}{\|u\|^{p-1}} J(u)$$
$$\geqslant J'(u;u) - k - k_1\frac{c_2\|u\| + c_1}{\|u\|^{p-1}}, \quad \forall u \in W^{2,p}(0,1),\ \|u\| \geqslant 1. \tag{49}$$

We are ready to prove that the sequence $\{u_n\}$ is bounded. Suppose on the contrary that $|u_n| \to \infty$. We may assume that $\|u_n\| \geqslant 1$ for all n. Then, applying (48) with $\rho = 1$ and (49) to (45) and (44), we get

$$pC + \frac{\tilde C}{\|u_n\|^{p-1}} \geqslant \frac{k_1}{p\|u_n\|^{p-1}} \int_0^1 \big(|u_n''|^p + a|u_n'|^p + b|u_n|^p\big)\,dt - \overline{m}_1 - k_1c_2\|u_n\|^{2-p} - \varepsilon_n\|u_n\|, \tag{50}$$

for some constants $\tilde C > 0$ and $\overline{m}_1 > 0$. Now, by $\lambda_1 > 0$ and Lemma 2.1, inequality (50) implies

$$pC + \tilde C + \overline{m}_1 \geqslant \left(\frac{k_1 m}{p} - \frac{k_1c_2}{\|u_n\|^{p-1}} - \varepsilon_n\right)\|u_n\|,$$

a contradiction since $\varepsilon_n \to 0$ and $\|u_n\| \to \infty$. Therefore $\{u_n\}$ is bounded.

Next, let $\{u_n\}$ again be a Palais–Smale sequence. Since $\{u_n\}$ is bounded under each of the hypotheses (G_θ) and (G_p), there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, and $u \in W^{2,p}(0,1)$, such that $u_n \rightharpoonup u$ in $W^{2,p}(0,1)$. Thus, $u_n \to u$ (for a subsequence) strongly in $C^1([0,1])$ and, since $D(j)$ is convex and closed, then $u \in D(J)$. From (18) we derive that

$$\Phi^0(u_n; u-u_n) + J'(u_n; u-u_n) + \varepsilon_n\|u-u_n\| \geqslant -\int_0^1 \big(|u_n''|^{p-2}u_n''u'' + |u_n'|^{p-2}u_n'u' + |u_n|^{p-2}u_nu\big)\,dt + \|u_n\|^p$$
$$\geqslant \|u_n\|^{p-1}\big(\|u_n\| - \|u\|\big),$$

yielding that

$$\Phi^0(u_n; u-u_n) + J'(u_n; u-u_n) + \varepsilon_n\|u-u_n\| - \|u\|^{p-1}\big(\|u_n\| - \|u\|\big)$$
$$\geqslant \big(\|u_n\|^{p-1} - \|u\|^{p-1}\big)\big(\|u_n\| - \|u\|\big). \tag{51}$$

Functional Φ has the trivial extension $\widetilde{\Phi}$ on the space $C^1([0,1])$ defined by (19). Moreover, $\widetilde{\Phi}$ is locally Lipschitz functional and obviously

$$\Phi^0(v;w) = \widetilde{\Phi}^0(v;w), \quad \forall v, w \in W^{2,p}(0,1).$$

The upper semi-continuity of $\widetilde{\Phi}^0(\cdot;\cdot)$ yields

$$\limsup_{n\to\infty} \Phi^0(u_n; u - u_n) \leqslant \widetilde{\Phi}^0(u;0) = 0. \tag{52}$$

On the other hand,

$$\begin{aligned}\limsup_{n\to\infty} J'(u_n; u - u_n) &\leqslant \limsup_{n\to\infty}\big(J(u) - J(u_n)\big)\\ &= J(u) - \liminf_{n\to\infty} J(u_n) \leqslant 0.\end{aligned} \tag{53}$$

Hence, taking into account (51)–(53), we obtain

$$0 \geqslant \limsup_{n\to\infty}\big(\|u_n\|^{p-1} - \|u\|^{p-1}\big)\big(\|u_n\| - \|u\|\big)$$

implying that $\|u_n\| \to \|u\|$. Since $(W^{2,p}(0,1), \|\cdot\|)$ is uniformly convex, $u_n \to u$ strongly in $W^{2,p}(0,1)$. Thus, I satisfies the Palais–Smale condition, as claimed. □

Proof of Theorem 1.2. We will show that functional I satisfies the hypotheses of the mountain pass Theorem 2.1. First of all, according to Lemma 2.3, it satisfies the Palais–Smale condition. Since $(0,0,0,0)^T \in \partial j((0,0,0,0)^T)$, we have

$$J(u) \geqslant J(0) = j\big((0,0,0,0)^T\big).$$

We assume without any loss of generality that $j((0,0,0,0)^T) = 0$ and so in particular $I(0) = 0$. Next, we will prove that there exist constants $\rho > 0$ and $\alpha > 0$ such that $I(u) \geqslant \alpha$ for all $u \in W^{2,p}(0,1)$ such that $\|u\| = \rho$. Here $\|\cdot\|$ denotes as usual the norm of $W^{2,p}(0,1)$. More precisely, there exist constants $\sigma > 0$ and $\delta > 0$, such that

$$F(t,x) \leqslant \frac{\lambda_1 - \sigma}{p}|x|^p, \quad \forall |x| \leqslant \delta. \tag{54}$$

Next, there exists a constant $d > 0$ such that $|u(t)| \leqslant d\|u\|$ for all $u \in W^{2,p}(0,1)$. Hence, if $\rho = \|u\| < d^{-1}\delta$, then inequality (54) can be applied with x replaced by $u(t)$. We have

$$\begin{aligned} I(u) &= \frac{1}{p}\int_0^1 \big(|u''|^p + a|u'|^p + b|u|^p\big)\,dt - \int_0^1 F(t,u)\,dt + J(u)\\ &\geqslant \frac{1}{p}\int_0^1 \big(|u''|^p + a|u'|^p + b|u|^p\big)\,dt - \frac{\lambda_1 - \sigma}{p}\int_0^1 |u|^p\,dt \geqslant m\|u\|^p, \end{aligned}$$

for some constant $m > 0$. Here, we have used the inequality

$$\int_0^1 \big(|u''|^p + a|u'|^p + (b - \lambda_1 + \sigma)|u|^p\big)\,dt \geqslant \sigma \int_0^1 |u|^p\,dt,$$

which is an immediate consequence of Lemma 2.1 for b replaced with $b - \lambda_1 + \sigma$.

Finally, we need to find a function $e \in W^{2,p}(0,1)$ such that

$$I(e) \leqslant 0 \quad \text{and} \quad \|e\| > \rho. \tag{55}$$

In what follows, we will examine separately the two alternative cases of the theorem.

Case 1. Let (G_θ) hold. Assume that $|x| > M$, where M is the constant which appear in the hypotheses of the theorem. The mapping $s \longmapsto s^{-\theta}F(t,sx)$ is locally Lipschitz for a.a. $t \in (0,1)$, so we have for each $s > 0$

$$\bar{\partial}_s\big(s^{-\theta}F(t,sx)\big) \subset \bar{\partial}_s\big(s^{-\theta}F(t,sx)\big) + s^{-\theta}\bar{\partial}_s\big(F(t,sx)\big) = s^{-\theta-1}\big(-\theta F(t,sx) + sx\bar{\partial}F(t,sx)\big).$$

Given $1 \leqslant r < s$, by Lebourg's mean value theorem and assumption (14), there exist $\tau \in (r,s)$ and $\xi \in \bar{\partial}_s(s^{-\theta}F(t,sx))|_{s=\tau}$, $\xi \geqslant 0$, such that

$$s^{-\theta}F(t,sx) - r^{-\theta}F(t,rx) = \xi(s-r) \geqslant 0,$$

i.e.,

$$F(t,sx) \geqslant s^\theta F(t,x), \quad \text{for a.a. } t \in [0,1],\ \forall |x| > M,\ s \geqslant 1.$$

Now, let $h \in C_0^\infty(0,1)$ be such that $|h| > M$ on a set with positive measure. Then,

$$\begin{aligned}\int_0^1 F(t,sh)\,dt &= \int_{\{|sh|>M\}} F(t,sh)\,dt + \int_{\{|sh|\leqslant M\}} F(t,sh)\,dt \\ &\geqslant \int_{\{|h|>M\}} F(t,sh)\,dt - M\int_0^1 \alpha_M(t)\,dt \\ &\geqslant s^\theta \int_{\{|h|>M\}} F(t,h)\,dt - M\int_0^1 \alpha_M(t)\,dt,\end{aligned}$$

for all $s \geqslant 1$. We have $J(sh) = 0$ for each s, thus

$$\begin{aligned}I(sh) &= \frac{s^p}{p}\int_0^1 \left(\left|h''\right|^p + a\left|h'\right|^p + b|h|^p\right)dt - \int_0^1 F(t,sh)\,dt \\ &\leqslant \frac{s^p}{p}\int_0^1 \left(\left|h''\right|^p + a\left|h'\right|^p + b|h|^p\right)dt - s^\theta \int_{\{|h|>M\}} F(t,h)\,dt + M\int_0^1 \alpha_M(t)\,dt\end{aligned}$$

for all $s \geqslant 1$. The latter inequality reads

$$I(sh) \leqslant s^p k_1 - s^\theta k_2 + k_3 \to -\infty, \quad \text{as } s \to \infty,$$

with constants $k_1, k_2, k_3 > 0$. Finally, take s_0 sufficiently large such that $I(s_0 h) \leqslant 0$ and $\|s_0 h\| > \rho$. Then $e := s_0 h$ satisfies conditions (55).

Case 2. Let (G_p) and $(\bar{L}_\infty)$ hold. Let $u_n \in \mathcal{D}$ and $s_n > 0$ be such that $\|u_n\|_{L^p} = 1$, $s_n \to \infty$,

$$\int_0^1 \left(\left|u_n''\right|^p + a\left|u_n'\right|^p + b|u_n|^p\right)dt + \frac{pJ(s_n u_n)}{s_n^p} \to \bar{\lambda}_1.$$

Condition $(\bar{L}_\infty)$ implies that there exist constants $C > 0$ and $\sigma > 0$ such that

$$F(t,x) \geqslant \frac{\bar{\lambda}_1 + \sigma}{p}|x|^p, \quad \forall |x| > C,\ \text{a.a. } t \in (0,1).$$

We have

$$\begin{aligned}\int_0^1 F\big(t, s_n u_n(t)\big)\,dt &= \int_{\{|s_n u_n|>C\}} F(t,s_n u_n)\,dt + \int_{\{|s_n u_n|\leqslant C\}} F(t,s_n u_n)\,dt \\ &\geqslant s_n^p \frac{\bar{\lambda}_1+\sigma}{p}\int_{\{|s_n u_n|>C\}} |u_n|^p\,dt - C\int_0^1 \alpha_C(t)\,dt \\ &= s_n^p \frac{\bar{\lambda}_1+\sigma}{p}\left(\int_0^1 |u_n|^p\,dt - \int_{\{|s_n u_n|\leqslant C\}} |u_n|^p\,dt\right) - C\int_0^1 \alpha_C(t)\,dt \\ &\geqslant s_n^p \frac{\bar{\lambda}_1+\sigma}{p} - \left|\frac{\bar{\lambda}_1+\sigma}{p}\right| C^p - C\int_0^1 \alpha_C(t)\,dt.\end{aligned}$$

Hence

$$I(s_nu_n) \leqslant \frac{s_n^p}{p}\int_0^1 \left(\left|u_n''\right|^p + a\left|u_n'\right|^p + b|u_n|^p\right)dt + J(s_nu_n) - s_n^p\frac{\bar{\lambda}_1+\sigma}{p} - \left|\frac{\bar{\lambda}_1+\sigma}{p}\right|C^p - C\int_0^1 \alpha_C(t)\,dt.$$

Therefore,

$$\frac{I(s_nu_n)}{s_n^p} \leqslant \frac{1}{p}\left(\int_0^1 \left(\left|u_n''\right|^p + a\left|u_n'\right|^p + b|u_n|^p\right)dt + p\frac{J(s_nu_n)}{s_n^p}\right) - \frac{(\bar{\lambda}_1+\sigma)}{p} - \frac{C_1}{s_n^p},$$

which converges to $-\sigma/p$ as $n \to \infty$. Finally, let n be such that $I(s_nu_n) < 0$ and $\|s_nu_n\| > \rho$. Obviously, $e := s_nu_n$ satisfies (55). □

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