An extension of the Jordan-von Neumann theorem

GHEORGHE MOROŞANU AND CRISTIAN VLADIMIRESCU

The purpose of this Note is to present an extension of the classical Jordan-von Neumann (JN) theorem [3] - which is recalled below - to the case of a normed space over the skew field $\mathbb{H}$ of quaternions. It is known that this extension is valid in a more general framework (see [5], [6], [7], [8]), but our approach is based on elementary arguments only. So, this result may be of interest for students, applied researchers, etc. We think this extension could be applied to control theory, mechanics and other areas.

Let us first recall (see, e.g., [1], p. 168, or [4], p. 244)

Theorem 1. (P. Jordan - J. von Neumann) *A real or complex normed space X, with the norm $\|\cdot\|$, is a pre-Hilbert space with the same norm if and only if the following parallelogram identity (PI) is satisfied*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right),$$

for every x, $y \in X$.

Recall also (see, e.g., [2], p. 67) that $\mathbb{H} = \{(a,b,c,d) \mid a,b,c,d \in \mathbb{R}\}$ is a skew field with respect to the following operations (addition and multiplication):

$$(a_1,b_1,c_1,d_1) + (a_2,b_2,c_2,d_2) := (a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2),$$

$$\begin{aligned}&(a_1,b_1,c_1,d_1)\cdot(a_2,b_2,c_2,d_2) := (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2,\\ &a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2,\ a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2,\\ &a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2).\end{aligned}$$

In other words, $(\mathbb{H}, +, \cdot)$ has an algebraic structure similar to a field except for commutativity of multiplication. For $\mathbf{v} = (a,b,c,d) \in \mathbb{H}$, we define the norm (absolute value)

$$|\mathbf{v}| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2},$$

and the conjugate

$$\bar{\mathbf{v}} = (a, -b, -c, -d)\,.$$

For $r \in \mathbb{R}$ and $\mathbf{v} = (a, b, c, d) \in \mathbb{H}$ let us denote as usual

$$r\mathbf{v} := (ra, rb, rc, rd)\,.$$

With the notations $\mathbf{1} = (1, 0, 0, 0)$, $\mathbf{i} = (0, 1, 0, 0)$, $\mathbf{j} = (0, 0, 1, 0)$, and $\mathbf{k} = (0, 0, 0, 1)$, we have

$$(a, b, c, d) = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \text{ for all } (a, b, c, d) \in \mathbb{H}.$$

We shall identify every real number a with the quaternion $(a, 0, 0, 0)$ and so $\mathbb{R}$ can be viewed as a subset of $\mathbb{H}$. So, every $(a, b, c, d) \in \mathbb{H}$ can be represented as $(a, b, c, d) = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$.

Note also that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},\ \mathbf{i}\cdot\mathbf{j} = -\mathbf{j}\cdot\mathbf{i} = \mathbf{k},\ \mathbf{j}\cdot\mathbf{k} = -\mathbf{k}\cdot\mathbf{j} = \mathbf{i},\ \mathbf{k}\cdot\mathbf{i} = -\mathbf{i}\cdot\mathbf{k} = \mathbf{j},$$

and

$$\overline{\mathbf{u}\cdot\mathbf{v}} = \bar{\mathbf{v}}\cdot\bar{\mathbf{u}},\quad |\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,,\quad \mathbf{u}\cdot\bar{\mathbf{u}} = |\mathbf{u}|^2\,,$$

for all $\mathbf{u},\ \mathbf{v} \in \mathbb{H}$.

Now, if X is a vector space over $\mathbb{H}$, one can define in a classical manner the concepts of norm, inner product, etc. Obviously, *the PI is still valid in a pre-Hilbert space over* $\mathbb{H}$*. Conversely, if X is a normed space over* $\mathbb{H}$*, with the norm* $\|\cdot\|$*, and the PI is satisfied, then we can define an inner product* $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{H}$, *by*

$$\begin{aligned}\langle x, y\rangle := \quad & \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 + \|x+\mathbf{i}y\|^2\,\mathbf{i} - \|x-\mathbf{i}y\|^2\,\mathbf{i} \\ & + \|x+\mathbf{j}y\|^2\,\mathbf{j} - \|x-\mathbf{j}y\|^2\,\mathbf{j} + \|x+\mathbf{k}y\|^2\,\mathbf{k} - \|x-\mathbf{k}y\|^2\,\mathbf{k}\Big)\,,\end{aligned}$$

for all $x,\ y \in X$, such that it generates the given norm $\|\cdot\|$. Let us prove that this mapping is indeed an inner product. First,

$$\begin{aligned}\overline{\langle y, x\rangle} &= \frac{1}{4}\Big(\|y+x\|^2 - \|y-x\|^2 - \|y+\mathbf{i}x\|^2\,\mathbf{i} + \|y-\mathbf{i}x\|^2\,\mathbf{i} \\ &\quad - \|y+\mathbf{j}x\|^2\,\mathbf{j} + \|y-\mathbf{j}x\|^2\,\mathbf{j} - \|y+\mathbf{k}x\|^2\,\mathbf{k} + \|y-\mathbf{k}x\|^2\,\mathbf{k}\Big) \\ &= \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 - \|x-\mathbf{i}y\|^2\,\mathbf{i} + \|x+\mathbf{i}y\|^2\,\mathbf{i} \\ &\quad - \|x-\mathbf{j}y\|^2\,\mathbf{j} + \|x+\mathbf{j}y\|^2\,\mathbf{j} - \|x-\mathbf{k}y\|^2\,\mathbf{k} + \|x+\mathbf{k}y\|^2\,\mathbf{k}\Big) \\ &= \langle x, y\rangle\,,\end{aligned}$$

for every $x,\ y \in X$. For $\mathbf{v} = (a,b,c,d) \in \mathbb{H}$ denote by $\mathrm{Re}\,\mathbf{v}$ the real number a. Since

$$\mathrm{Re}\ \langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right),$$

we have

$$\begin{aligned}
\mathrm{Re}\ \langle \mathbf{i}x,y\rangle &= \frac{1}{4}\left(\|\mathbf{i}x+y\|^2 - \|\mathbf{i}x-y\|^2\right)\\
&= \frac{1}{4}\left(\|x-\mathbf{i}y\|^2 - \|x+\mathbf{i}y\|^2\right),\\
\mathrm{Re}\ \langle \mathbf{j}x,y\rangle &= \frac{1}{4}\left(\|x-\mathbf{j}y\|^2 - \|x+\mathbf{j}y\|^2\right),\\
\mathrm{Re}\ \langle \mathbf{k}x,y\rangle &= \frac{1}{4}\left(\|x-\mathbf{k}y\|^2 - \|x+\mathbf{k}y\|^2\right).
\end{aligned}$$

Therefore,

$$\langle x,y\rangle = \mathrm{Re}\ \langle x,y\rangle - (\mathrm{Re}\ \langle \mathbf{i}x,y\rangle)\,\mathbf{i} - (\mathrm{Re}\,\langle \mathbf{j}x,y\rangle)\,\mathbf{j} - (\mathrm{Re}\ \langle \mathbf{k}x,y\rangle)\,\mathbf{k}.$$

This together with the identity (whose proof is similar to that in [1], p. 169, or [4], p. 245)

$$\mathrm{Re}\ \langle x,y\rangle + \mathrm{Re}\ \langle z,y\rangle = \mathrm{Re}\ \langle x+z,y\rangle$$

implies

$$\langle x,y\rangle + \langle z,y\rangle = \langle x+z,y\rangle,$$

for all $x,y,z \in X$. In a standard manner, we infer that

$$\langle rx,y\rangle = r\,\langle x,y\rangle, \quad \text{for every } r \in \mathbb{R} \text{ and } x,\ y \in X.$$

In order to extend this property to all $r \in \mathbb{H}$ (i.e., $\langle rx,y\rangle = r\cdot\langle x,y\rangle$, for all $r \in \mathbb{H}$ and $x,\ y \in X$), it is sufficient to remark that

$$\begin{aligned}
\langle \mathbf{i}x,y\rangle &= \frac{1}{4}\Big(\|\mathbf{i}x+y\|^2 - \|\mathbf{i}x-y\|^2 + \|\mathbf{i}x+\mathbf{i}y\|^2\,\mathbf{i} - \|\mathbf{i}x-\mathbf{i}y\|^2\,\mathbf{i}\\
&\quad + \|\mathbf{i}x+\mathbf{j}y\|^2\,\mathbf{j} - \|\mathbf{i}x-\mathbf{j}y\|^2\,\mathbf{j} + \|\mathbf{i}x+\mathbf{k}y\|^2\,\mathbf{k} - \|\mathbf{i}x-\mathbf{k}y\|^2\,\mathbf{k}\Big)\\
&= \mathbf{i}\cdot\frac{1}{4}\Big(-\|x-\mathbf{i}y\|^2\,\mathbf{i} + \|x+\mathbf{i}y\|^2\,\mathbf{i} + \|x+y\|^2 - \|x-y\|^2\\
&\quad - \|x-\mathbf{k}y\|^2\,\mathbf{k} + \|x+\mathbf{k}y\|^2\,\mathbf{k} + \|x+\mathbf{j}y\|^2\,\mathbf{j} - \|x-\mathbf{j}y\|^2\,\mathbf{j}\Big)\\
&= \mathbf{i}\cdot\langle x,y\rangle,
\end{aligned}$$

and, similarly, $\langle \mathbf{j}x, y\rangle = \mathbf{j} \cdot \langle x, y\rangle$, $\langle \mathbf{k}x, y\rangle = \mathbf{k} \cdot \langle x, y\rangle$, for every $x,\ y \in X$. Finally,

$$\langle x, x\rangle = \|x\|^2\,, \text{ for every } x \in X,$$

which completes the proof. □

Remark 1. $\mathbb{H}$ *is a Hilbert space over itself, where the inner product is defined by* $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u} \cdot \bar{\mathbf{v}}$. *Another typical example of a Hilbert space over* $\mathbb{H}$ *is* $L^2(\Omega, \mathbb{H})$, *with the inner product*

$$\langle \mathbf{f}, \mathbf{g}\rangle = \int_\Omega \mathbf{f}(\xi) \cdot \overline{\mathbf{g}(\xi)}\, d\xi,$$

where Ω *is an open subset of* $\mathbb{R}^k$.

Remark 2. *An immediate consequence of the above result is the following: A Banach space* X *over* $\mathbb{H}$ *is a Hilbert space with the same norm if and only if the PI is fulfilled in* X.

Acknowledgements. Many thanks are due to D. Bakic (University of Zagreb, Croatia), P. Saworotnow (The Catolic University of America, Washington, D.C., USA), and B. Zalar (University of Ljubljana, Slovenia) for interesting and useful discussions concerning this Note.

References

[1] R. Cristescu, *Analiză funcţională*, Ediţia a II-a, Editura Didactică şi Pedagogică, Bucureşti, 1970.

[2] I.D. Ion, N. Radu, *Algebră*, Ediţia a III-a, Editura Didactică şi Pedagogică, Bucureşti, 1981.

[3] P. Jordan, J. von Neumann, On inner products in linear metric spaces, *Annals of Mathematics* **36**, 719-723 (1935).

[4] M. Schechter, *Principles of Functional Analysis*, Academic Press, New York and London, 1971.

[5] P. Šemrl, On quadratic and sesquilinear functionals, *Aequationes Mathematicae*, **31**, 184-190 (1986).

[6] P. Šemrl, On Jordan $*-$derivations and an application, *Colloquium Mathematicum*, **LIX** (3), 241-251 (1990).

[7] P. Šemrl, Quadratic functionals and Jordan $*-$derivations, *Studia Mathematica*, **97** (3), 157-165 (1991).

[8] B. Zalar, Jordan-von Neumann theorem for Saworotnow's generalized Hilbert space, *Acta Mathematica Hungarica*, **69**(4), 301-325 (1995).

Gheorghe MOROŞANU
Department of Mathematics and Its Applications
Central European University
Nador u. 9, H-1051 Budapest, Hungary
E-mail: morosanug@ceu.hu

Cristian VLADIMIRESCU
Department of Mathematics
University of Craiova
13 Al.I. Cuza Str., Craiova RO-200585, Romania
E-mail: vladimirescucris@hotmail.com