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A Variational Approach to a Problem Arising in Capillarity Theory

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This paper studies a model for capillarity in circular tubes. The main result of this paper states the existence of a unique C^1 solution for this model. This solution is a minimum point of some functional Ψ. © 1997 Academic Press

1. INTRODUCTION

Consider the problem

$$(tG(x'(t)))' = tx(t), \qquad 0 \le t \le 1, \tag{1}$$

$$x'(1) = \beta, \tag{2}$$

where β is a positive constant and G satisfies the following assumptions:

(a) G: $[0, \beta] \to \mathbb{R}$ is continuous, strictly increasing, and $G(0) = 0$.

Note that in the particular case $G(u) = \mu u(1 + u^2)^{-1/2}$, $\mu > 0$, problem (1), (2) represents a model for capillarity in circular tubes [1, 4]. See also [2, pp. 289–293].

Problem (1), (2) was solved by A. Corduneanu and G. Moroşanu [1, 4] under more restrictive assumptions by using a direct method. Our aim here is to derive the same result by a variational approach.

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As we shall see later, it is convenient to extend G outside the interval $[0,\beta]$ by linear functions. For example, let $G_1\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$G_1(u) = \begin{cases} u, & \text{for } u < 0, \\ G(u), & \text{for } 0 \le u \le \beta, \\ u - \beta + G(\beta), & \text{for } u > \beta. \end{cases}$$

Obviously $G_1 \in C(\mathbb{R})$ and G_1 is strictly increasing. We shall prove that the equation

$$\big(tG_1(x'(t))\big)' = tx(t), \qquad 0 \le t \le 1 \tag{3}$$

with boundary condition (2) has a unique solution $x \in C^1[0, 1]$ with $0 \le x'(t) \le \beta$ for $0 \le t \le 1$ and hence x is also a solution of problem (1), (2).

Denoting $j(u) := \int_0^u G_1(s)\,ds$ we can expect to obtain the solution of (3), (2) in $C^1[0,1]$ as a minimum point of the functional

$$\Psi(v) = \int_0^1 t\{j(v'(t)) + v^2(t)/2\}\,dt - G(\beta)v(1). \tag{4}$$

2. AUXILIARY RESULTS

Before stating and proving the main result we give two auxiliary results.

LEMMA 1. *If assumptions* (A) *are satisfied then for each* $\beta > 0$, *problem* (3), (2) *has at most one solution in* $C^1[0,1]$.

Proof. Let $x_1, x_2 \in C^1[0,1]$ be solutions of (3), (2). We have

$$\int_0^1 (x_1 - x_2)\big(t(G_1(x_1') - G_1(x_2'))\big)'\,dt = \int_0^1 t(x_1 - x_2)^2\,dt.$$

Integrating by parts in the left hand side and using the monotonicity of G_1 and the fact that $x_1'(1) = x_2'(1) = \beta$ we can easily see that $x_1 = x_2$ in $[0,1]$.

Q.E.D.

LEMMA 2. *If assumptions* (A) *are satisfied, and* $x \in C^1[0,1]$ *is a solution of Eq.* (3), *then the following implications hold*:

$$x(0) = 0 \Rightarrow \text{ either } x'(t) \ge 0 \text{ for } 0 \le t \le 1 \text{ or } x'(t) \le 0 \text{ for } 0 \le t \le 1; \tag{5}$$

$$x(0) > 0 \Rightarrow x'(t) > 0 \qquad \text{for } 0 < t \le 1; \tag{6}$$

$$x(0) < 0 \Rightarrow x'(t) < 0 \qquad \text{for } 0 < t \le 1. \tag{7}$$

Proof. Multiplying $x(t)$ by Eq. (3) and then integrating on $[0, t]$ we get

$$tx(t)G_1(x'(t)) = \int_0^t s\{x^2(s) + x'(s)G_1(x'(s))\}\,ds, \qquad 0 \le t \le 1. \quad (8)$$

Therefore, taking into account the properties of G_1, we can see that

$$\{t \in (0,1]\,;\, x(t) = 0\} = \{t \in (0,1]\,;\, x'(t) = 0\}$$

and this set is either an empty set or an interval of the form $(0, \delta]$. Using this remark and Eq. (8) we can easily derive the conclusions (5), (6), and (7). Q.E.D.

3. THE MAIN RESULT

THEOREM 1. *If assumptions* (A) *are satisfied then for each* $\beta > 0$ *problem* (1), (2) *has a unique solution in* $C^1[0, 1]$.

Proof. As uniqueness is already proved (see Lemma 1 which is also valid for problem (1), (2)) it remains to prove existence. To do that we shall use the functional Ψ defined by (4). Consider the space

$$H = \{v = v(t)\,;\, t^{1/2}v,\, t^{1/2}v' \in L^2(0,1)\}.$$

This is a Hilbert space with scalar product

$$\langle v_1, v_2 \rangle_H = \int_0^1 t(v_1 v_2 + v_1' v_2')\,dt.$$

Now, since

$$r^2/3 - C \le j(r) \le r^2 + C, \qquad r \in \mathbb{R}, \quad (9)$$

where C is a positive constant, we can see that Ψ is everywhere defined on H and moreover it is coercive,

$$\Psi(v) \ge C_1\|v\|_H^2 - C_2, \qquad v \in H, \quad (10)$$

where C_1, C_2 are positive constants. Indeed for $v \in H$ and δ fixed in $(0,1)$ we have

$$|v(1)| \le C_3\left(\|v\|^2_{L^2(\delta,1)} + \|v'\|^2_{L^2(\delta,1)}\right)^{1/2}$$

and so

$$|v(1)| \le C_4\left(\|t^{1/2}v\|^2_{L^2(\delta,1)} + \|t^{1/2}v'\|^2_{L^2(\delta,1)}\right)^{1/2} \le C_4\|v\|_H.$$

This implies (10) by a straightforward computation. Since Ψ is convex, continuous, and coercive it has a minimum point $x \in H$ (see, e.g., [3, p. 34]). Therefore we have (Euler–Lagrange equation)

$$\left(tG_1(x'(t))\right)' = tx(t), \qquad \text{for a.e. } t \in (0,1). \tag{11}$$

For any $\delta \in (0,1)$ we can deduce from (11) and $x \in H$ that the function $f(t) := tG_1(x'(t))$ belongs to $C^1[\delta,1]$ and so $x' \in C[\delta,1]$ because $G_1^{-1} \in C(\mathbb{R})$. Hence $x \in C^1(0,1]$. As x is a minimum point of ψ we also have

$$G_1(x'(1)) = G_1(\beta). \tag{12}$$

Hence x verifies (2) and

$$tG_1(x'(t)) = G(\beta) - \int_t^1 sx(s)\,ds, \qquad 0 < t \leq 1. \tag{13}$$

From (13) we deduce that there exists

$$\lim_{t\to 0^+} tG_1(x'(t)) = l \in \mathbb{R}.$$

By the definition of G_1 it follows that

$$l = \lim_{t\to 0^+} tx'(t).$$

In fact $l = 0$ because otherwise

$$t^2(x'(t))^2 \geq l^2/2 > 0$$

in some interval $(0,\delta)$ and this contradicts the fact that $x \in H$. Now, it is easy to deduce from (11) that x satisfies

$$tG_1(x'(t)) = \int_0^t sx(s)\,ds, \qquad 0 \leq t \leq 1. \tag{14}$$

In the next step we shall prove that $x \in C^1[0,1]$. We have

$$\left|\int_0^t sx(s)\,ds\right| \leq 2^{-1/2}t\left(\int_0^t sx^2(s)\,ds\right)^{1/2}, \qquad 0 \leq t \leq 1.$$

Hence, by (14) we can see that $G_1(x'(t)) \to 0$ as $t \to 0^+$ and this implies $x'(t) \to 0$ as $t \to 0^+$. Therefore $x \in C^1[0,1]$, $x'(0) = 0$ and x verifies (3) for all $t \in [0,1]$.

The final step is to prove that x is a solution for Eq. (1). To this end we first note that $\beta > 0$ implies (see Lemma 2)

$$x'(t) \geq 0 \qquad \text{and} \qquad x(t) \geq 0 \text{ for } 0 \leq t \leq 1. \tag{15}$$

Moreover, using (15) we can easily see that the function $t \mapsto (1/t) \int_0^t sx(s)\,ds$ is nondecreasing in [0, 1]. Since

$$x'(t) = G_1^{-1}\left(\frac{1}{t}\int_0^t sx(s)\,ds\right) \tag{16}$$

and G_1^{-1} is nondecreasing it follows that x' is also nondecreasing in [0, 1]. Therefore $0 = x'(0) \le x'(t) \le x'(1) = \beta$ for $0 \le t \le 1$ which implies that x is a solution of Eq. (1). Q.E.D.

4. FINAL COMMENTS

We have incidentally proved some properties of the solution of problem (1), (2) with physical significance (see [2–4] for the description of the model arising in capillarity theory): $x'(0) = 0$, (15), and the fact that x' is nondecreasing (that is, x is a convex function).

In order to point out some other properties of solutions let us assume in what follows that $G \in C(\mathbb{R})$, $G(0) = 0$, and G is strictly increasing. By the above reasoning we can see that for any $\beta \in \mathbb{R}$ problem (1), (2) has a unique solution, say $x = x(t, \beta)$.

Now, if x_1, x_2 are solutions of Eq. (1) we have

$$\begin{aligned} t[x_1(t) - x_2(t)][G(x_1'(t)) - G(x_2'(t))] \\ = \int_0^t s\left\{(x_1 - x_2)^2 + (x_1' - x_2')[G(x_1') - G(x_2')]\right\} ds, \quad 0 \le t \le 1. \end{aligned} \tag{17}$$

This implies that

$$\{t \in (0,1];\ x_1(t) = x_2(t)\} = \{t \in (0,1];\ x_1'(t) = x_2'(t)\} \tag{18}$$

and this set is either a void set or an interval $(0, \delta]$. Therefore, we have the monotonicity property

$$\beta_1 < \beta_2 \Rightarrow x'(t, \beta_1) \le x'(t, \beta_2) \qquad \text{for } 0 \le t \le 1. \tag{19}$$

Using (17), (18), and (19) we can easily deduce that

$$\beta_1 < \beta_2 \Rightarrow x(t, \beta_1) \le x(t, \beta_2) \qquad \text{for } 0 \le t \le 1. \tag{20}$$

As an immediate consequence of (19) and (20) we can show the continuity of the solution of β. More precisely, using Arzelà's Criterion and Eq. (1) we can see that $\beta_n \to \beta$ implies $x(\cdot, \beta_n) \to x(\cdot, \beta)$ in $C^1[0, 1]$.

On the other hand, if x_1, x_2 are solutions of Eq. (1) with $x_1(0) < x_2(0)$ then $x_1(t) < x_2(t)$ for $0 \le t \le 1$ and $x_1'(t) < x_2'(t)$ for $0 < t \le 1$ (see (18)).

Let us also mention that in the case in which we have uniqueness for the Cauchy problem associated to Eq. (1) (according to (16), this happens if G_1^{-1} is locally Lipschitz), then making again use of (18) we can see that $\beta_1 < \beta_2 \Rightarrow x'(t, \beta_1) < x'(t, \beta_2)$ for $0 < t \le 1$ and $x(t, \beta_1) < x(t, \beta_2)$ for $0 \le t \le 1$.

In particular, for $\beta > 0$, $x'(t, \beta) > 0$ in $(0, 1]$ and $x(t, \beta) > 0$ in $[0, 1]$. Moreover, in this case, it follows by (16) that x' is strictly increasing (i.e., x is strictly convex).

However, in general, that uniqueness property does not hold as the following simple counterexample shows. Let $G \in C(\mathbb{R})$ be a strictly increasing function such that $G(u) = u^2/45$ for $0 \le u \le 3$. Then Eq. (1) with the initial condition $x(0) = 0$ has at least two solutions: $x_1(t) = 0$ and $x_2(t) = t^3$.

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