Math. Sci. Res. Hot-Line 3 (1999), No. 4, 1-22

A CLASS OF NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS*

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Abstract. A class of nonlinear parabolic boundary value problems (1.1) is studied by means of the theory of abstract Cauchy problems containing time dependent maximal monotone and compact operators whose domains may depend on time.

Key words. Nonlinear parabolic boundary value problem, maximal monotone operator, subdifferential, abstract Cauchy problem.
AMS Subject Classifications: 35K55, 34G20, 47H15.

1. Introduction

We denote $u_x = \partial u/\partial x$, $u_t = \partial u/\partial t$ and study the problem

$$u_t(t,x) - w_x(t,x) + \hat{w}(t,x) = f(t,x), \text{ for a.e. } x \in]0,1[, t \in]0,T[, \tag{1.1a}$$

$$w(t,x) \in G(t,x)u_x(t,x), \text{ for a.e. } x \in]0,1[, t \in]0,T[, \tag{1.1b}$$

$$\hat{w}(t,x) \in K(t,x)u(t,x), \text{ for a.e. } x \in]0,1[, t \in]0,T[, \tag{1.1c}$$

$$(w(t,0), -w(t,1)) \in \beta(t)(u(t,0), u(t,1)), \text{ for a.e. } t \in]0,T[, \tag{1.1d}$$

$$u(0,x) = u_0(x), \text{ for a.e. } x \in]0,1[, \tag{1.1e}$$

where $T > 0$ and $G(t,x), K(t,x) \subset \mathbf{R} \times \mathbf{R}$ and $\beta(t) \subset \mathbf{R}^2 \times \mathbf{R}^2$ are maximal monotone operators. This very general model describes heat conduction and diffusion phenomena. For G independent on t and linear in u_x, see **[M1]**, **[MP]** (see also **[M2]** for time dependent boundary conditions). Equations with nonlinear terms in u_x are investigated in **[GL]**, **[L]**, **[LF]** (see also their references). Equations like (1.1) in more dimensions are considered in **[H]**, **[T]**, ...

Here we study (1.1) by using abstract Cauchy problems (1.2) in a real Hilbert space with maximal monotone operators $A(t)$, perturbed by operators $B(t)$;

$$u'(t) + A(t)u(t) + B(t)u(t) \ni f(t),\ t > 0,\ u(0) = u_0. \tag{1.2}$$

In Section 3 we consider (1.1) as (1.2) with time independent $A(t)$ and $B(t) \equiv 0$. In Section 4 we study the existence of a solution for (1.2). Our condition (H.3) allowing

*This research is supported by The Academy of Finland and by The Romanian Academy.

$A(t)$ to be a subdifferential with time dependent domain, is close to that of **[AB]**; our proofs are based on the methods used for the case of time independent $A(t)$ in **[Br]**; the perturbed problems are handled by fixed point theorems. In Section 5 our results will be applied to (1.1) with unbounded $\beta(t)$.

2. The notation

Let H be a real Hilbert space, $\|\cdot\|_H$ its norm and $(\cdot,\cdot)_H$ its inner product. An operator $A\subset H\times H$ is *monotone* if $(y_2-y_1,x_1-x_2)_H\geq 0$, for each $(x_1,y_1),(x_2,y_2)\in A$. It is *maximal monotone* if it is not contained by any other monotone operator of H. Let $\lambda>0$, $\psi: H\mapsto]-\infty,\infty]$ be convex and A be maximal monotone in H. The *subdifferential* of ψ is denoted by $\partial\psi$ and ψ_λ is the *Yosida Moreau regularization* of ψ. The *resolvent* and the *Yosida approximate* of A are denoted by J_λ and by A_λ, respectively; A^0x is the element of Ax with the minimal norm. Indeed,

$$\partial\psi=\{(u,v)\in H\times H\mid \psi(u)<\infty,\ \psi(u)+(v,\xi-u)_H\leq\psi(\xi)\ \text{for each}\ \xi\in H\},$$
$$J_\lambda=(I+\lambda A)^{-1},\ A_\lambda=\frac{1}{\lambda}(I-J_\lambda),\ \psi_\lambda(x)=\inf\{\frac{1}{2\lambda}\|y-x\|_H^2+\psi(y)\mid y\in H\}.$$

The positive part of ψ is denoted by ψ_+. For the further details and the theory of differential equations containing maximal monotone operators, we refer to **[Br]**, **[Ba]**, **[BP]** and **[M1]**. For the general functional analysis, the reader may see **[KA]**.

3. The case of time independent G and K

In this section we study problem (3.1) with non-homogenous boundary conditions;

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}G(x,u_x)+K(x,u)=f(t,x),\ 0<x<1,\ \ t>0, \tag{3.1a}$$
$$s(t)+\big(G(0,u_x(t,0)),-G(1,u_x(t,1))\big)\in\beta\big(u(t,0),u(t,1)\big),\ t>0, \tag{3.1b}$$
$$u(0,x)=u_0(x),\ 0<x<1. \tag{3.1c}$$

Let us introduce our main assumptions. Let $\delta>0$.

(I_1) $G\in C^1([0,1]\times\mathbf{R})$ and $G_\xi(x,\xi)\geq\delta$, for each $x\in[0,1]$, $\xi\in\mathbf{R}$.

(I_2) $K\in C([0,1]\times\mathbf{R})$ and $K(x,\cdot)$ is monotone, for each $x\in[0,1]$.

(I_3) $\beta\subset\mathbf{R}^2\times\mathbf{R}^2$ is a maximal monotone operator.

(I_4) $j:\mathbf{R}^2\mapsto]-\infty,\infty]$ is proper, convex, and lower semicontinuous; $\beta=\partial j$.

It is well known **[M1]** that (3.1b) includes many classical boundary value conditions: Dirichlet, Neumann, Robin-Steklov, periodic, etc.

Define $H=L^2(0,1)$, $u(t)=u(t,\cdot)$, $f(t)=f(t,\cdot)$ and $A(t):D(A(t))\mapsto H$,

$$D(A(t))=\{v\in H^2(0,1)\,|\,s(t)+(G(0,v'(0)),-G(1,v'(1))\in\beta(v(0),v(1))\}, \tag{3.2a}$$
$$A(t)v(x)=-\frac{d}{dx}G(x,v'(x))+K(x,v(x)),\ 0<x<1,\ t\geq 0. \tag{3.2b}$$

If $s \equiv 0$, we denote $A = A(t)$. Now, (3.1) is a Cauchy problem (3.3);

$$u'(t) + A(t)u(t) \ni f(t),\ t > 0,\ u(0) = u_0. \tag{3.3}$$

Proposition 3.1. Assume I_1, I_2 and I_4. Let $s \equiv 0$. Then $\phi : H \mapsto [-\infty, \infty]$,

$$\phi(v) = \begin{cases} \int_0^1 \big(g(\cdot, v') + k(\cdot, v)\big)\,dx + j(v(0), v(1)), \text{ if } v \in H^1(0,1), \\ \infty, \text{ otherwise}, \end{cases} \tag{3.4a}$$

$$g(x,\xi) = \int_0^\xi G(x,\tau)\,d\tau,\ k(x,\xi) = \int_0^\xi K(x,\tau)\,d\tau, \tag{3.4b}$$

is proper, convex, and lower semicontinuous, $A = \partial\phi$, and A is densely defined.
Proof. By I_1, $G(0,\cdot)$ and $G(1,\cdot)$ are surjective. So, there exist $a, b, c, d \in \mathbf{R}$ such that $(G(0,c), -G(1,d)) \in \beta(a,b)$. Therefore $\emptyset \neq D(A)$, since it contains $\hat{v}$,

$$\hat{v}(x) = (2a - 2b + c + d)x^3 + (-3a + 3b - 2c - d)x^2 + cx + a. \tag{3.5}$$

Hence $\{\hat{v} + \psi \mid \psi \in C_0^\infty(]0,1)[\} \subset D(A)$. Thus $D(A)$ is dense in H.

By Lemma 5.1 below, ϕ is proper, convex and lower semicontinuous.

Clearly, $A \subset \partial\phi$. Let us show that $\partial\phi \subset A$. So, let $(u,v) \in \partial\phi$. Then by Lemma 5.1 below, $u \in H^1(0,1)$ and there is $w \in H^1(0,1)$ such that

$$v(x) = -w'(x) + K(x, u(x)) = (Au)(x),\ \text{for a.e. } x \in]0,1[, \tag{3.6a}$$

$$w(x) = G(x, u'(x)),\ \text{for a.e. } x \in]0,1[, \tag{3.6b}$$

$$(w(0), -w(1)) \in \beta(u(0), u(1)). \tag{3.6c}$$

By I_1 and by the Implicit Function Theorem, $x \mapsto G(x,\cdot)^{-1}\xi$ belongs to $C^1([0,1])$, for each $\xi \in \mathbf{R}$. Since $G(x,\cdot)^{-1}$ are $\frac{1}{\delta}$-Lipschitzian and $w \in H^1(0,1)$, then $x \mapsto u'(x) = G(x,\cdot)^{-1}w(x)$ belongs to $H^1(0,1)$. Hence $u \in H^2(0,1)$. Proposition 3.1 is proved.
Proposition 3.2. Assume I_1, I_2 and I_3. Then $A(t)$ given by (3.2) is a maximal monotone operator.
Proof. By Remark 5.2 below, the operator given by (5.2) is maximal monotone; we denote it by $\tilde{A}(t)$. Clearly, $A(t) \subset \tilde{A}(t)$. Let $v \in H$. Then there are $u, w \in H^1(0,1)$ satisfying (3.6). Thus $u \in H^2(0,1)$ and $A(t) = \tilde{A}(t)$.
Remark 3.1. Propositions 3.1 and 3.2 improve the results of **[GL]** and of **[AM]**.
Theorem 3.1. (Existence and uniqueness if $\beta = \partial j$). Assume I_1, I_2 and I_4, let $T > 0$, $u_0 \in H$, $f \in L^2(Q_T)$, $Q_T =]0,T[\times]0,1[$ and $s \in \mathbf{R}^2$. Then (3.1) has a unique strong solution $u \in C([0,T];H)$ with $(t,x) \mapsto \sqrt{t}u_t(t,x)$ in $L^2(Q_T)$. If, in addition, $u_0 \in D(\phi)$, then $u \in H^1(Q_T)$. If, in addition, $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$, then $u \in L^\infty(0,T;H^1(0,1)) \cap L^\infty(Q_T)$.

Proof. The first part is implied by [**Br**, Thm. 3.6] (see also Proposition 3.1 above). If $u_0 \in D(\phi)$, we have from the same theorem that $u_t \in L^2(Q_T)$, so $Au = f - u_t \in L^2(Q_T)$. Let $\hat{v}$ be given by (3.5). By I_2, for a.e. $t \in]0, T[$,

$$\delta \int_0^1 \big(u_x(t,x) - \hat{v}'(x)\big)^2 \, dx \leq \int_0^1 \Big(f(t,x) - u_t(t,x) - A\hat{v}(x)\Big)\big(u(t,x) - \hat{v}(x)\big)\, dx. \quad (3.6)$$

Hence $u \in H^1(Q_T)$. Finally, if $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;L^2(0,1))$, then $u_t \in L^\infty(0,T;H)$ (see [**M1**, p. 48]). So, using (3.6), we can see that $u_x \in L^\infty(0,T;H)$ and hence $u \in L^\infty(0,T;H^1(0,1))$. Then $u \in L^\infty(Q_T)$, by

$$u(t,x) = \int_0^1 \Big(\tau \frac{\partial u}{\partial \tau}(t,\tau) + u(t,\tau)\Big)\, d\tau - \int_x^1 \frac{\partial u}{\partial \tau}(t,\tau) d\tau. \quad (3.7)$$

Theorem 3.1 is proved.

Remark 3.2. If I_1, I_2 and I_4 are fulfilled, $u_0 \in H$ and $f \in L^1(0,T;H)$, then problem (3.1) has a unique weak solution.

Proposition 3.3. If I_1, I_2 and I_3 are satisfied and if $s(t) \equiv 0$, then the resolvent of A, $J_\lambda = (I + \lambda A)^{-1} : H \mapsto H$, is compact, for every $\lambda > 0$.

Proof. Let $\lambda > 0$ and $Y \subset H$ be bounded. Denote $u_p = (I + \lambda A)^{-1} p$, for each $p \in Y$. Let $\hat{v}$ be given by (3.5). Then there is $M' > 0$, independent on p;

$$M' + \frac{1}{2}\|u_p - \hat{v}\|_H^2 \geq (p - \hat{v} - \lambda A\hat{v}, u_p - \hat{v})_H \geq \|u_p - \hat{v}\|_H^2 + \lambda\delta\|u_p' - \hat{v}'\|_H^2.$$

Hence the resolvent is bounded as $H \mapsto H^1(0,1)$, so it is compact as $H \mapsto H$.

Theorem 3.2. (Asymptotic behaviour if $\beta = \partial j$). Assume I_1, I_2 and I_3. Let $u_0 \in H$, $f \in L^1(0,\infty;H)$, $s(t) \equiv 0$, $F := A^{-1}0 \neq \emptyset$, and let u be the weak solution of (3.1). Then there exists $\hat{p} \in F$ such that $u(t) \to \hat{p}$ in H, as $t \to \infty$. If, in addition, $f \in W^{1,1}(0,\infty;H)$, then $u(t) \to \hat{p}$ weakly in $H^1(0,1)$ and strongly in $C([0,1])$, as $t \to \infty$.

Proof. For $f \in W^{1,1}(0,\infty;H)$ it follows by Proposition 3.3 that the trajectory $\{u(t) \mid t \geq \epsilon\}$ is bounded in $H^1(0,1)$, for each $\epsilon > 0$. Indeed,

$$u(t) = (I + A)^{-1}\big(f(t) + u(t) - \frac{d^+u}{dt}(t)\big) \quad (3.8)$$

and the set $\{u(t) \mid t \geq 0\}$ is bounded in H, because $F \neq \emptyset$ (see [**M1**, p. 73]). For the rest of the proof see [**M1**, Chapters II, III].

Remark 3.3. For β not subdifferential, but $s(t) \equiv 0$, and $G(x, \cdot)$ linear, see [**M1**] and [**MP**]. Still A is maximal monotone in H.

Next, we study nonhomogenuous boundary conditions. As both G and β are nonlinear, (3.1c) cannot be homogenized. So (3.1) has time dependent $D(A(t))$.

We recall that $u \in C([0,T];H)$ is said to be a *weak solution* of (3.3) on $[0,T]$ (cf. [**Br**, p. 64]), if there exist $u_{n0} \in H$, $f_n \in L^1(0,T;H)$, maximal monotone $A_n(t) \subset H \times H$ and $u_n \in W^{1,\infty}(0,T;H)$, $n = 1,2,\ldots$, satisfying

$$u_n'(t) + A_n(t)u_n(t) \ni f_n(t), \text{ for a.e. } t \in]0,T[, \ u_n(0) = u_{n0}, \tag{3.9}$$

$$u_n \to u \text{ in } C([0,T];H) \text{ and } f_n \to f \text{ in } L^1(0,T;H), \text{ as } n \to \infty. \tag{3.10}$$

Theorem 3.3. (Existence of weak solutions). Assume I_1, I_2, and I_3. Let $u_0 \in H$, $f \in L^1(0,T;H)$, and $s \in L^2(0,T;\mathbf{R}^2)$. Then (3.3) has a unique weak solution.
Proof. Let $u_{n0} \in D(A)$, $f_n \in W^{1,\infty}(0,T;H)$, $s_n \in W_0^{1,\infty}(0,T;\mathbf{R}^2)$, $\tilde{A}_n(t) = A_n(t) - f_n(t)$, where $A_n(t)$ is $A(t)$ given by (3.2) with $s_n(t)$ instead of $s(t)$. We assume that

$$u_{n0} \to u_0 \text{ in } H, \ s_n \to s \text{ in } L^2(0,T;\mathbf{R}^2), \ f_n \to f \text{ in } L^1(0,T;H), \text{ as } n \to \infty. \tag{3.11}$$

By Prop. 3.2 each $\tilde{A}_n(t)$ and $A_n(t)$ is maximal monotone. We recall that

$$\|(\xi(0),\xi(1))\|_{\mathbf{R}^2} \le 2\|\xi'\|_H + 2\|\xi\|_H, \text{ for each } \xi \in H^1(0,1). \tag{3.12}$$

Let $\sigma, \tau \in [0,T]$, $v \in D(\tilde{A}_n(\tau))$, and $w \in D(\tilde{A}_n(\sigma))$. By (3.2), I_1 and by (3.12),

$$-(v-w, \tilde{A}_n(\tau)v - \tilde{A}_n(\sigma)w)_H \le -\big(G(x,v'(x)) - G(x,w'(x))\big)\big(v(x)-w(x)\big)\Big|_0^1 -$$

$$-\delta\|v'-w'\|_H^2 + \cdots \le 2\|v-w\|_H^2 + (\tau-\sigma)^2\Big(\frac{1+\delta}{\delta}\|s_n'\|^2_{L^\infty(0,T;\mathbf{R}^2)} + \|f_n'\|^2_{L^\infty(0,T;H)}\Big).$$

Hence we obtain from [**T**, pp. 147, 138-9, 142-3] that the problems (3.9) have the solution $u_n \in W^{1,\infty}(0,T;H)$. By (3.9), (3.2), I_1 and by (3.12), for a.e. $t \in]0,T[$,

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)-u_m(t)\|_H^2 + \frac{\delta}{2}\|u_{nx}(t)-u_{mx}(t)\|_H^2 \le \frac{2}{\delta}\|s_n(t)-s_m(t)\|^2_{\mathbf{R}^2} +$$

$$+ \|u_n(t)-u_m(t)\|_H\Big(2\|s_n(t)-s_m(t)\|_{\mathbf{R}^2} + \|f_n(t)-f_m(t)\|_H\Big). \tag{3.13}$$

We integrate and apply a Gronwall type inequality [**Br**, p. 157]. Then

$$\|u_n(t)-u_m(t)\|_H \le \|u_{n0}-u_{m0}\|_H + \frac{2}{\sqrt{\delta}}\|s_n-s_m\|_{L^2(0,T;\mathbf{R}^2)} +$$

$$+ 2\|s_n-s_m\|_{L^1(0,T;\mathbf{R}^2)} + \|f_n-f_m\|_{L^1(0,T;H)}, \tag{3.14}$$

for each $t \in [0,T]$, $m,n = 1,2,\ldots$ This completes the proof.
Remark 3.4. By (3.13), the weak solution $u \in C([0,T];H) \cap L^2(0,T;H^1(0,1))$.

Theorem 3.4. (Asymptotic behaviour). Assume I_1, I_2, I_3, and let $u_0 \in H$, $f \in L^1(0,\infty;H)$, $s \in L^2(0,\infty;\mathbf{R}^2) \cap L^1(0,\infty;\mathbf{R}^2)$, $u \in L^2_{loc}(]0,\infty[;H^1(0,1)) \cap C([0,\infty[;H)$, where u is the weak solution of (3.3). Let all the weak solutions of (3.3) with $f \equiv 0$, $s \equiv 0$ converge in H, as $t \to \infty$. Then $u(t) \to p \in A^{-1}0$ in H, as $t \to \infty$.

Proof. Let u_n be the weak solution of (3.3) with (f_n, s_n) instead of (f, s),

$$f_n(t) = \begin{cases} f(t), & \text{if } 0 < t < n, \\ 0, & \text{if } t \geq n, \end{cases} \qquad s_n(t) = \begin{cases} s(t), & \text{if } 0 < t < n, \\ 0, & \text{if } t \geq n, \end{cases} \qquad n = 1, 2, \ldots$$

We have (3.14) for the approximates of (u_n, f_n, u_0) and of (u, f, u_0). Taking the limit we obtain, for each $t > 0$ and $n = 1, 2, \ldots$,

$$\|u_n(t) - u(t)\|_H \leq \frac{2}{\sqrt{\delta}} \|s\|_{L^2(n,\infty;\mathbf{R}^2)} + 2\|s\|_{L^1(n,\infty;\mathbf{R}^2)} + \|f\|_{L^1(n,\infty;H)}. \tag{3.15}$$

There is $p_n \in H$ such that $u_n(t) \to p_n$ in H, as $t \to \infty$. By (3.14), (p_n) is a Cauchy sequence, converging in H toward some $p \in H$. Then by (3.15),

$$\|u(t) - p\|_H \leq \|u_n(t) - p_n\|_H + \|u(t) - u_n(t)\|_H + \|p_n - p\|_H < \epsilon + \|u_n(t) - p_n\|_H,$$

for any $\epsilon > 0$ if n is large enough. Therefore,

$$\limsup_{t \to \infty} \|u(t) - p\|_H \leq \epsilon, \text{ for each } \epsilon > 0.$$

Hence $u(t) \to p$ in H, as $t \to \infty$. Theorem 3.4 is proved.

4. Non-autonomous Cauchy problem

Let H be a real Hilbert space, $M, T > 0$ and $\eta \in L^1(0,T)$. For $\phi : [0,T] \times H \mapsto]-\infty,\infty]$ and $B(t) \subset H \times H$, $t \in [0,T]$, we formulate:

(H.1) For each $t \in [0,T]$, $\phi(t,\cdot)$ is a proper convex lower semicontinuous function and $B(t)$ is a maximal monotone operator in H.

(H.2) There is $z \in L^2(0,T;H)$ such that $z(t) \in D(\partial\phi(t,\cdot))$ for a.e. $t \in]0,T[$, and $t \mapsto \phi(t, z(t))$, $t \mapsto \|\partial\phi^0(t,\cdot)z(t)\|_H^2$ and $t \mapsto \|B^0(t)z(t)\|_H^2$ are integrable.

(H.3) For each $\lambda \in]0,1]$ and $y \in H$, $\phi_\lambda(\cdot, y) \in W^{1,1}(0,T)$ and a.e on $]0,T[$,

$$\frac{\partial \phi_\lambda}{\partial t}(t,y) \leq \frac{1}{3}\|\partial\phi_\lambda(t,\cdot)y\|_H^2 + M\|y\|_H^2 + \eta(t)\Big(1 + \phi_{\lambda+}(t,y)\Big) \geq \|B_\lambda(t)y\|_H^2.$$

(H.4) For each $y \in H$ and $\lambda \in]0,1]$, the mappings $t \mapsto (I + \lambda\partial\phi(t,\cdot))^{-1}y$ and $t \mapsto (I + \lambda B(t))^{-1}y$ are measurable.

Theorem 4.1. Assume (H.1)-(H.4). If $u_0 \in D(\phi(0,\cdot))$ and $f \in L^2(0,T;H)$, then there exist v, $w \in L^2(0,T,H)$, a unique $u \in H^1(0,T;H)$, and some constants M_1, $M_2 > 0$, independent on f and u_0, such that

$$u'(t) + v(t) + w(t) = f(t), \text{ for a.e. } t \in]0,T[, \tag{4.1a}$$
$$v(t) \in \partial\phi(t,\cdot)u(t), \; w(t) \in B(t)u(t), \text{ for a.e. } t \in]0,T[, \tag{4.1b}$$
$$u(0) = u_0; \tag{4.1c}$$

$$\|u\|^2_{L^\infty(0,T;H)} + \int_0^T \Big(\|u'(\tau)\|^2_H + \|v(\tau)\|^2_H + \|w(\tau)\|^2_H\Big)\,d\tau + \operatorname*{ess\,sup}_{\tau\in[0,T]} \phi_+(\tau,u(\tau)) \\ \le M_1 \int_0^T \|f(\tau)\|^2_H\,d\tau + M_2\big(1 + \|u_0\|^2_H + \phi_+(0,u_0)\big). \tag{4.2}$$

Theorem 4.2. Assume (H.1)-(H.4). Let $u_0 \in \overline{D(\phi(0,\cdot))}$, $f \in L^1(0,T;H)$ and

$$M_f = \int_0^T \tau\|f(\tau)\|^2_H\,d\tau + \|f\|^2_{L^1(0,T;H)}, \; M_{u_0} = \int_0^T \frac{1}{\tau}\|z(\tau) - u_0\|^2_H\,d\tau + \|u_0\|^2_H,$$

be finite. Then there exist measurable v, $w : [0,T] \mapsto H$, a unique $u \in C([0,T];H)$, differentiable a.e. on $]0,T[$, and constants M_3, $M_4 > 0$, independent on u_0 and f. They satisfy (4.1) and (4.3);

$$\|u\|^2_{L^\infty(0,T;H)} + \int_0^T \tau\Big(\|u'(\tau)\|^2_H + \|v(\tau)\|^2_H + \|w(\tau)\|^2_H\Big)\,d\tau + \operatorname*{ess\,sup}_{\tau\in[0,T]} \tau\phi_+(\tau,u(\tau)) \\ \le M_3 M_f + M_4 M_{u_0} + M_4. \tag{4.3}$$

We state for $C(t) : D(C(t)) \mapsto H$, $D(C(t)) \subset H$, $t \in [0,T]$, $\alpha \ge 0$, $\beta > 0$:

(H.5) For each $y \in C([0,T];H)$ with $y(t) \in D(C(t))$, $t \in [0,T]$, the mappings $t \mapsto C(t)y(t)$ and $t \mapsto t^\alpha\|C(t)y(t)\|^2_H$ are integrable.

(H.6) For a.e. $t \in]0,T[$, $D(C(t)) = \overline{D(\partial\phi(t,\cdot))}$, and

$$\|C(t)x - C(t)y\|_H \le \eta(t)\|x - y\|_H, \text{ for each } x, y \in D(C(t)).$$

(H.7) The closure of $\{y \in H \mid \|y\|^2_H + \phi(t,y) \le \delta, \text{ for a.e. } t \in [0,T]\}$ is compact in H, for each $\delta > 0$.

(H.8) There is $\psi : [0,T] \times H \mapsto [0,\infty]$, measurable with respect to the σ-field generated by the products of Lebesgue sets in $[0,T]$ and of Borel sets in H. For a.e. $t \in]0,T[$ and each $y \in D(\partial\phi(t,\cdot))$, $\psi(t,\cdot)$ is proper, convex and lower semicontinuous, $D(\psi(t,\cdot)) \subset D(C(t))$, and

$$\beta\|C(t)y\|^2_H - T^{-1-\alpha}\|y\|^2_H - \eta(t) \le \psi(t,y) \le \|\partial\phi^0(t,\cdot)y\|^2_H + \|B^0(t)y\|^2_H + \eta(t).$$

(H.9) $y \mapsto Cy$, $(Cy)(t) = C(t)y(t)$, is strongly-weakly closed in $L^2(0,T;H)$.
(H.10) For a.e. $t \in]0,T[$ and for each $y \in D(\partial\phi(t,\cdot))$,

$$\|C(t)y\|_H^2 \leq \frac{t\eta(t)\psi(t,y)}{\beta\|\eta\|_{L^1(0,T)}} + \frac{1}{\beta}\Big(\frac{\eta(t)\|x\|_H}{\|\eta\|_{L^1(0,T)}}\Big)^2 + \eta(t)^2.$$

Theorem 4.3. Assume (H.5) with $\alpha = 0$, the conditions of Theorem 4.1, and, in addition, (H.6) or (H.7)-(H.9) with $\beta = 3M_1$. Then there exist $u \in H^1(0,T;H)$ and $v, w \in L^2(0,T;H)$ which satisfy (4.4);

$$u'(t) + v(t) + w(t) + C(t)u(t) = f(t), \text{ for a.e. } t \in]0,T[, \tag{4.4a}$$

$$v(t) \in \partial\phi(t,\cdot)u(t),\ w(t) \in B(t)u(t), \text{ for a.e. } t \in]0,T[, \tag{4.4b}$$

$$u(0) = u_0. \tag{4.4c}$$

If, in addition, (H.6) is satisfied, then u is unique.
Theorem 4.4. Assume (H.5) with $\alpha = 1$, the conditions of Theorem 4.2, and, in addition, (H.6) or (H.7)-(H.10) with $\beta = 4M_3$. Then there exist $u \in C([0,T];H)$, differentiable a.e. on $]0,T[$, and measurable $v, w : [0,T] \mapsto H$ which satisfy (4.4) and

$$\int_0^T \tau\Big(\|u'(\tau)\|_H^2 + \|v(\tau)\|_H^2 + \|w(\tau)\|_H^2\Big)\, d\tau < \infty.$$

If, in addition, (H.6) is satisfied, then u is unique.
Remark 4.1. Condition (H.3) allows the domain of $\partial\phi(t,\cdot)$ to depend on time. Indeed, let $H = \mathbf{R}$, $c_1, c_2 \in H^1(0,1)$ with $c_1 \leq c_2$ and $\phi : [0,1] \times \mathbf{R} \mapsto \{0,\infty\}$,

$$\phi(t,x) = \begin{cases} 0, & \text{if } c_1(t) \leq x \leq c_2(t), \\ \infty, & \text{otherwise.} \end{cases}$$

Then, for each $x \in \mathbf{R}$ and for a.e. $t > 0$, $\phi_\lambda(\cdot,x) \in W^{1,1}(0,1)$ and

$$\frac{\partial\phi_\lambda}{\partial t}(t,x) \leq \frac{1}{3}|\partial\phi_\lambda(t,\cdot)x|^2 + \frac{3}{4}\max(c_1'(t)^2, c_2'(t)^2).$$

Remark 4.2. The estimates for (u,v,w) in Theorems 4.3 and 4.4 can be obtained from (4.2) and (4.3) by substituting f by $f - Cu$.
Remark 4.3. The coefficient 1/3 in (H.3) can be slightly increased, depending on $B(t)$. If $B(t) \equiv 0$, then 1/3 can be replaced by any $\alpha \in]0,1[$.
Proof of Theorem 4.1. We modify a text book proof for the case of autonomous equation, see [**M1**, pp. 46-61], cf. [**Br**, pp. 54-57, 72-78]. We denote constants, independent on f, λ, t and u_0, by $M_5, M_6, \ldots$

Let $\lambda \in]0,1]$. The mappings $x \mapsto \partial\phi_\lambda(t,\cdot)x + B_\lambda(t)x$, $H \mapsto H$, $t \in [0,T]$, are Lipschitzian and $t \mapsto \partial\phi_\lambda(t,\cdot)x + B_\lambda(t)x$, $[0,T] \mapsto H$, $x \in H$, are square integrable. Thus by [**Br**, p. 10], (4.5) has a unique solution $u_\lambda \in H^1(0,T;H)$;

$$u'_\lambda(t) + \partial\phi_\lambda(t,\cdot)u_\lambda(t) + B_\lambda(t)u_\lambda(t) = f(t), \text{ for a.e. } t \in]0,T[, \tag{4.5a}$$
$$u_\lambda(0) = u_0. \tag{4.5b}$$

Lemma 4.1. Let $\delta \in [0,T[$. The realizations of $\partial\phi(t,\cdot)$ and of $B(t)$, i.e.

$$\partial\phi = \{(x,y) \in L^2(0,T;H)^2 \mid y(t) \in \partial\phi(t,\cdot)x(t), \text{ for a.e. } t \in]0,T[\},$$
$$B = \{(x,y) \in L^2(0,T;H)^2 \mid y(t) \in B(t)x(t), \text{ for a.e. } t \in]0,T[\}$$

are maximal monotone in $L^2(\delta,T;H)$.
Proof. The monotonicity of B and of $\partial\phi$ is clear. The maximality is implied by $R(I + B) = R(I + \partial\phi) = L^2(\delta,T;H)$, which holds since $t \mapsto (I + \partial\phi(t,\cdot))^{-1}y(t)$ and $t \mapsto (I + B(t))^{-1}y(t)$ are square integrable, for each $y \in L^2(\delta,T;H)$.
Lemma 4.2. There are M_1, $M_2 > 0$, independent on λ, f and on u_0, such that

$$\|u_\lambda\|^2_{L^\infty(0,T;H)} + \int_0^T \Big(\|u'_\lambda(\tau)\|^2_H + \|\partial\phi_\lambda(\tau)u_\lambda(\tau)\|^2_H + \|B_\lambda(\tau)u_\lambda(\tau)\|^2_H\Big)\,d\tau +$$
$$+ \sup_{\tau\in[0,T]} \phi_{\lambda+}(\tau,u_\lambda(\tau)) \le M_1 \int_0^T \|f(\tau)\|^2_H\,d\tau + M_2\big(1 + \|u_0\|^2_H + \phi_+(0,u_0)\big).$$

Proof. Since ϕ_λ, u_λ and $\partial\phi_\lambda u_\lambda$ satisfy the conditions of chain rule [**H**, Lemma 6], then $\phi_\lambda(\cdot,u_\lambda(\cdot)) \in W^{1,1}(0,T)$ and, for a.e. $t \in]0,T[$,

$$\frac{d}{dt}\phi_\lambda(t,u_\lambda(t)) = (\partial\phi_\lambda(t,\cdot)u_\lambda(t), u'_\lambda(t))_H + \frac{\partial\phi_\lambda}{\partial t}(t,u_\lambda(t)). \tag{4.6}$$

We multiply (4.5a) by $u'_\lambda(t)$, by $100\partial\phi_\lambda(t,\cdot)u_\lambda(t)$, and by $B_\lambda(t)u_\lambda(t)$, successively. By summing the results, by (4.6) and (H.3),

$$\frac{1}{303}\|u'_\lambda(t)\|^2_H + \frac{d}{dt}\phi_\lambda(t,u_\lambda(t)) + \frac{1}{22}\|\partial\phi_\lambda(t,\cdot)u_\lambda(t)\|^2_H + \frac{1}{101}\|B_\lambda(t)u_\lambda(t)\|^2_H$$
$$\le 25\|f(t)\|^2_H + 3\eta(t)\big(1 + \phi_{\lambda+}(t,u_\lambda(t))\big) + 3M\|u_\lambda(t)\|^2_H, \tag{4.7}$$

for a.e. $t \in]0,T[$, whence by integrating and by $\phi_\lambda \le \phi \le \phi_+$ [**Br**, p. 39],

$$\int_0^t \Big(\frac{1}{303}\|u'_\lambda(\tau)\|^2_H + \frac{1}{22}\|\partial\phi_\lambda(\tau,\cdot)u_\lambda(\tau)\|^2_H + \frac{1}{101}\|B_\lambda(\tau)u_\lambda(\tau)\|^2_H\Big)\,d\tau +$$
$$+ \phi_\lambda(t,u_\lambda(t)) \le \phi(0,u_0) + 25\|f\|^2_{L^2(0,T;H)} + 3\|\eta\|_{L^1(0,T)} +$$
$$+ 3M\int_0^t \|u_\lambda(\tau)\|^2_H\,d\tau + \int_0^t 3\eta(\tau)\frac{1}{2}\big(1 + \operatorname{sgn}\phi_\lambda(\tau,u_\lambda(\tau))\big)\phi_\lambda(\tau,u_\lambda(\tau))\,d\tau. \tag{4.8}$$

By Gronwall's inequality [**Br**, p. 156], there is a constant $M_5 > 0$ such that

$$\phi_\lambda(t, u_\lambda(t)) \leq M_5\Big(1 + \phi_+(0, u_0) + \|f\|^2_{L^2(0,T;H)} + \int_0^t \|u_\lambda(\tau)\|^2_H \, d\tau\Big), \tag{4.9}$$

for each $t \in [0, T]$. By (4.8) and by (4.9), for each $t \in [0, T]$,

$$\begin{aligned} \|u_\lambda(t)\|^2_H \leq 2\|u_0\|^2_H + 606T\frac{1}{303}\int_0^t \|u'_\lambda(\tau)\|^2_H \, d\tau \leq 2\|u_0\|^2_H - 606T\phi_\lambda(t, u_\lambda(t)) \\ + M_6\Big(1 + \phi_+(0, u_0) + \|f\|^2_{L^2(0,T;H)} + \int_0^t \|u_\lambda(\tau)\|^2_H \, d\tau\Big). \end{aligned} \tag{4.10}$$

By the definition for subdifferential and by $\phi_\lambda(t, \cdot) \geq \phi(t, J_\lambda(t)\cdot)$ [**Br**, p. 39],

$$\begin{aligned} \phi_\lambda(t, x) &\geq \phi(t, J_\lambda(t)z(t)) + (\partial\phi_\lambda(t, \cdot)z(t), x - z(t))_H \\ &\geq -\frac{1}{909T}\|x\|^2_H + \phi(t, z(t)) - (228T + 1)\|\partial\phi^0(t, \cdot)z(t)\|^2_H - \|z(t)\|^2_H, \end{aligned} \tag{4.11}$$

for each $t \in [0, T]$ and $x \in H$. By (4.10), by (4.11), by integrating and by Gronwall's inequality [**Br**, p. 156],

$$\int_0^T \|u_\lambda(t)\|^2_H \, dt \leq M_8\Big(1 + \|u_0\|^2_H + \phi_+(0, u_0) + \|f\|^2_{L^2(0,T;H)}\Big). \tag{4.12}$$

Since $\phi(T, \cdot)$ is a proper convex lower semicontinuous function, it is bounded from below by an affine function [**Br**, p. 25]. Since there is $\tilde{z} \in D(\partial\phi(T, \cdot))$ and $\phi(T, J_\lambda(T)\cdot) \leq \phi_\lambda(T, \cdot)$, we have, for any $x \in H$,

$$-\phi_\lambda(T, x) \leq M_8\|J_\lambda(T)x\|_H + M_8 \leq \frac{1}{909T}\|x\|^2_H + M_9. \tag{4.13}$$

We obtain now Lemma 4.2 from (4.8)-(4.10) and from (4.12).
Lemma 4.3. There are a subsequence of (λ) tending to $0+$, $v, w \in L^2(0, T; H)$, and $u \in H^1(0, T; H)$ such that $u'_\lambda \to u'$, $\partial\phi_\lambda u_\lambda \to v$, $B_\lambda u_\lambda \to w$ weakly in $L^2(0, T; H)$ and $u_\lambda \to u$ in $C([0, T]; H)$, as $\lambda \to 0+$.
Proof. By Lemma 4.2, there are $u^*, v, w \in L^2(0, T; H)$ and a subsequence with $u'_\lambda \to u^*$, $\partial\phi_\lambda u_\lambda \to v$ and $B_\lambda u_\lambda \to w$ weakly in $L^2(0, T; H)$.

Let $\lambda, \mu \in]0, 1]$. By (4.5a) and by the monotonicity of $\partial\phi_\lambda(t, \cdot)$ and of $B_\lambda(t)$,

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|u_\lambda(t) - u_\mu(t)\|^2_H \leq (\lambda + \mu)\Big(2\|\partial\phi_\lambda(t, \cdot)u_\lambda(t)\|^2_H + 2\|\partial\phi_\mu(t, \cdot)u_\mu(t)\|^2_H\Big) + \\ + (\lambda + \mu)\Big(2\|B_\lambda(t)u_\lambda(t)\|^2_H + 2\|B_\mu(t)u_\mu(t)\|^2_H\Big), \text{ for a.e. } t \in]0, T[, \end{aligned}$$

(see [**Br**, p. 56]), whence by integrating, by (4.5b) and by Lemma 4.2,

$$\|u_\lambda - u_\mu\|^2_{C([0,T];H)} \le 8(\lambda+\mu)\Big(M_1\|f\|^2_{L^2(0,T;H)} + M_2\big(1+\|u_0\|^2_H + \phi(0,u_0)\big)\Big).$$

Hence (u_λ) converges toward some $u \in C([0,T];H)$. Since the derivative is a strongly-weakly closed mapping in $L^2(0,T;H)$, $u' = u^*$. Thus $u \in H^1(0,T;H)$.

By Lemmas 4.2 and 4.3 and by the definition of the Yosida approximate,

$$\|u - J_\lambda u_\lambda\|_{L^2(0,T;H)} \le \|u - u_\lambda\|_{L^2(0,T;H)} + \lambda\|\partial\phi_\lambda u_\lambda\|_{L^2(0,T;H)} \to 0, \text{ as } \lambda \to 0+\,.$$

Since $\partial\phi_\lambda u_\lambda \in \partial\phi J_\lambda u_\lambda$, $\partial\phi_\lambda u_\lambda \to v$ weakly in $L^2(0,T;H)$ and $\partial\phi$ is a maximal monotone operator, the demiclosedness result [**Br**, p. 27] implies $v \in \partial\phi u$. Similarly, $w \in Bu$. Hence (u,v,w) is a solution for (4.1). Lemmas 4.2 and 4.3 and the weak lower semicontinuity of the norm of $L^2(0,T;H)$ and of $\phi(t,\cdot)$ imply (4.2). Indeed, for a subsequence,

$$\phi(t,u(t)) \le \liminf_{\lambda\to 0+} \phi(t, J_\lambda(t)u_\lambda(t)) \le \liminf_{\lambda\to 0+} \phi_\lambda(t,u_\lambda(t)), \text{ for a.e. } t \in]0,T[.$$

Let (u,v,w), $(\tilde u,\tilde v,\tilde w) \in H^1(0,T;H)\times L^2(0,T;H)^2$ be two solutions of (4.1). Then by (4.1a) and by the monotonicity of $\partial\phi(t,\cdot)$ and of $B(t)$,

$$\frac{d}{dt}\frac{1}{2}\|u(t)-\tilde u(t)\|^2_H = (u(t)-\tilde u(t), -v(t)-w(t)+\tilde v(t)+\tilde w(t))_H \le 0,$$

for a.e. $t \in]0,T]$, whence by integrating over $[0,t] \subset [0,T]$ and by (4.1c), $u \equiv \tilde u$.

Theorem 4.1 is proved.

Proof of Theorem 4.2. Let $u_{0n} \in D(\phi(0,\cdot))$ and $f_n \in L^2(0,T;H)$,

$$\|u_0 - u_{0n}\|_H \le \frac{1}{n},\ f_n(t) = \begin{cases} f(t), & \text{if } \|f(t)\|_H \le n, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } n = 1,2,\ldots$$

By Theorem 4.1, there are $u_n \in H^1(0,T;H)$, $v_n, w_n \in L^2(0,T;H)$ satisfying

$$u_n'(t) + v_n(t) + w_n(t) = f_n(t), \text{ for a.e. } t \in]0,T[, \tag{4.14a}$$

$$v_n(t) \in \partial\phi(t,\cdot)u_n(t),\ w_n(t) \in B(t)u_n(t), \text{ for a.e. } t \in]0,T[, \tag{4.14b}$$

$$u_n(0) = u_{0n}. \tag{4.14c}$$

Lemma 4.4. There are some constants $M_3, M_4 > 0$, independent on f and u_0 and satisfying (4.3) with (u_n,v_n,w_n) instead of (u,v,w).

Proof. By the proof of Theorem 4.1 there are solutions $(u_{n\lambda})$ of (4.5) with (u_{0n}, f_n) instead of (u_0,f), satisfying Lemma 4.2 and converging in the sense of Lemma 4.3 toward (u_n,v_n,w_n), a solution of (4.14).

By the definition for subdifferential, by the modified (4.5a), by $\phi_\lambda \le \phi$, and by the monotonicity of $B_\lambda(t)$, for a.e. $t \in]0,T]$,

$$\begin{aligned}\phi_\lambda(t,u_{n\lambda}(t)) &\le (f_n(t) - B_\lambda(t)u_{n\lambda}(t) - u'_{n\lambda}(t), u_{n\lambda}(t) - z(t))_H + \phi_\lambda(t,z(t)) \\ &\le \Big(\|f(t)\|_H + \|B^0(t)z(t)\|_H\Big)\|u_{n\lambda}(t) - u_0\|_H - \frac{d}{dt}\frac{1}{2}\|u_{n\lambda}(t) - u_0\|_H^2 + \\ &+ \frac{t}{2}\|f(t)\|_H^2 + \frac{t}{2}\|B^0(t)z(t)\|_H^2 + \frac{t}{606}\|u'_{n\lambda}(t)\|_H^2 + \frac{152}{t}\|u_0 - z(t)\|_H^2 + \phi(t,z(t)).\end{aligned}$$

We multiply the modified (4.7) by t and integrate it over $[0,t] \subset [0,T]$. Then

$$\begin{aligned}&\frac{1}{606}\int_0^t \tau\Big(\|u'_{n\lambda}(\tau)\|_H^2 + \|\partial\phi_\lambda(\tau,\cdot)u_{n\lambda}(\tau)\|_H^2 + \|B_\lambda(\tau)u_{n\lambda}(\tau)\|_H^2\Big)\,d\tau + t\phi_\lambda(t,u_{n\lambda}(t)) + \\ &\quad + \frac{1}{2}\|u_{n\lambda}(t) - u_0\|_H^2 \le M_{11}\int_0^t \big(\|f(\tau)\|_H + \|u_{n\lambda}(\tau)\|_H\big)\|u_{n\lambda}(\tau) - u_0\|_H\,d\tau + \\ &+ M_{11}(1 + M_f + M_{u_0}) + \int_0^t 3\eta(\tau)\frac{1}{2}\big(1 + \operatorname{sgn}\phi_\lambda(\tau,u_{n\lambda}(\tau))\big)\tau\phi_\lambda(\tau,u_{n\lambda}(\tau))\,d\tau, \qquad (4.15)\end{aligned}$$

for each $t \in [0,T]$. By Gronwall's inequality [**Br**, p. 156], for each $t \in [0,T]$,

$$\begin{aligned}t\phi_\lambda(t,u_{n\lambda}(t)) &\le M_{12}\Big(1 + M_f + M_{u_0}\Big) + \\ &\qquad + M_{12}\int_0^t \Big(\|f(\tau)\|_H + \|u_{n\lambda}(\tau)\|_H\Big)\|u_{n\lambda}(\tau) - u_0\|_H\,d\tau. \qquad (4.16)\end{aligned}$$

By the modified (4.5) and by the monotonicity of $B_\lambda(t)$ and of $\partial\phi_\lambda(t,\cdot)$,

$$\begin{aligned}\frac{1}{2}\|u_{n\lambda}(t) - u_0\|_H^2 &= \frac{1}{2}\|u_{0n} - u_0\|_H^2 + \int_0^t (u_{n\lambda}(\tau) - u_0, u'_{n\lambda}(\tau))_H\,d\tau \\ &\le \frac{1}{2} + \int_0^t \Big(\frac{\tau}{2}\|u'_{n\lambda}(\tau)\|_H^2 + \frac{1}{\tau}\|z(\tau) - u_0\|_H^2 + \|f(\tau)\|_H\|u_{n\lambda}(\tau) - u_0\|_H + \\ &\quad + \frac{\tau}{2}\|f(\tau)\|_H^2 - (u_{n\lambda}(\tau) - z(\tau), \partial\phi_\lambda(\tau,\cdot)z(\tau) + B_\lambda(\tau)z(\tau))_H\Big)\,d\tau \\ &\le M_{13} + M_f + M_{u_0} + \int_0^T \frac{\tau}{2}\|u'_{n\lambda}(\tau)\|_H^2\,d\tau + \int_0^t \tilde\eta(\tau)\|u_{n\lambda}(\tau) - u_0\|_H\,d\tau,\end{aligned}$$

where $\tilde\eta \in L^1(0,1)$. By the Gronwall type inequality, (4.15-16) and by (4.11),

$$\begin{aligned}\|u_{n\lambda}(t)\|_H^2 &\le M_{14} + 4M_f + 4M_{u_0} + 2\int_0^T \tau\|u'_{n\lambda}(\tau)\|_H^2\,d\tau \\ &\le M_{15} + 4M_f + 4M_{u_0} + \frac{1}{2}\|u_{n\lambda}(T)\|_H^2, \text{ for each } t \in [0,T].\end{aligned}$$

By (4.16), (4.15) and by (4.11), $(u_{n\lambda}, \partial\phi_\lambda u_{n\lambda}, B_\lambda u_{n\lambda})$ satisfies (4.3). By the weak lower semicontinuity of the norms and by $\liminf_{\lambda\to 0+} \phi_\lambda(t, u_{n\lambda}(t)) \geq \phi(t, u_n(t))$, (4.3) is satisfied also by $(u_n, v_n, w_n, \phi(u_n))$.

Lemma 4.5. Denote $u_n^*(t) = \sqrt{t}\, u_n'(t)$, $\hat{v}_n(t) = \sqrt{t}\, v_n(t)$ and $\hat{w}_n(t) = \sqrt{t}\, w_n(t)$, for each $t \in [0,T]$. There are $u \in C([0,T];H)$, $u^*, \hat{v}, \hat{w} \in L^2(0,T;H)$ such that, on subsequences, as $n \to \infty$,

$$u_n^* \to u^*,\ \hat{v}_n \to \hat{v},\ \hat{w}_n \to \hat{w} \text{ weakly in } L^2(0,T;H) \text{ and } u_n \to u \text{ in } C([0,T];H).$$

Proof. The weak limits hold by the boundedness of $(\hat{u}_n^*)$, $(\hat{v}_n)$ and of $(\hat{w}_n)$. By (4.5a) and by the monotonicity of $\partial\phi(t,\cdot)$ and of $B(t)$,

$$\frac{d}{dt}\frac{1}{2}\|u_n(t) - u_m(t)\|_H^2 \leq \|u_n(t) - u_m(t)\|_H \|f_n(t) - f_m(t)\|_H, \text{ for a.e. } t \in]0,T[,$$

whence by integrating, by (4.5b) and by the Gronwall type inequality,

$$\|u_n - u_m\|_{C([0,T];H)} \leq \|u_{0n} - u_{0m}\|_H + \int_0^T \|f_n(t) - f_m(t)\|_H\, dt.$$

Thus (u_n) is a Cauchy sequence, converging toward $u \in C([0,T];H)$.

Lemma 4.6. Denote $v(t) = \hat{v}(t)/\sqrt{t}$, $w(t) = \hat{w}(t)/\sqrt{t}$, for each $t \in]0,T]$. Then u is differentiable and $u'(t) = u^*(t)/\sqrt{t}$, $v(t) \in \partial\phi(t,\cdot)u(t)$, $w(t) \in B(t)u(t)$ a.e. on $]0,T[$.

Proof. Let $\delta \in]0,T[$ and $\xi \in C_0^\infty(]\delta,T[)$. Then, as $n \to \infty$,

$$\int_\delta^T u^*(t)\frac{\xi(t)}{\sqrt{t}}\, dt \leftarrow \int_\delta^T u_n^*(t)\frac{\xi(t)}{\sqrt{t}}\, dt = -\int_\delta^T u_n(t)\xi'(t)\, dt \to -\int_\delta^T u(t)\xi'(t)\, dt.$$

Thus $u \in H^1(\delta,T;H)$ and $u'(t) = u^*(t)/\sqrt{t}$, for a.e. $t \in]\delta,T[$. Since $u_n \to u$ strongly and $v_n \to v$, $w_n \to w$ weakly in $L^2(\delta,T;H)$, the demiclosedness result of maximal monotone operators and Lemma 4.1 give that $v \in \partial\phi u$ and $w \in Bu$.

By the weak lower semicontinuity of the norms and of $\phi(t,\cdot)$ and by Lemmas 4.4 and 4.5, (u,v,w) satisfy (4.3) and (4.1). The uniqueness in u of the solution can be proved as in the proof of Theorem 4.1. Theorem 4.2 is proved.

Proof of Theorems 4.3 and 4.4. Assume (H.6). (Cf. [**A**, Prop. 4.2]). Define a complete metric space X with the norm $\|\cdot\|_{C([0,T];H)}$,

$$X = \{y \in C([0,T];H) \mid y(t) \in D(C(t)),\ \text{for each } t \in [0,T]\}.$$

Define $\mathcal{T} : X \mapsto X$, $\mathcal{T}x = u$, where $u \in C([0,T];H)$ is a solution of

$$u'(t) + v(t) + w(t) = f(t) - C(t)x(t),\ \text{for a.e. } t \in]0,T], \tag{4.17a}$$

$$v(t) \in \partial\phi(t,\cdot)u(t),\ w(t) \in B(t)u(t),\ \text{for a.e. } t \in]0,T[, \tag{4.17b}$$

$$u(0) = u_0. \tag{4.17c}$$

By Theorem 4.1 or 4.2, such $u \in X$ exists and is unique, for each $x \in X$.

Let $x, y \in X$ and $k = 1, 2, \dots$ By (4.17), by (H.6) and by the monotonicities,

$$\frac{d}{dt}\frac{1}{2}\|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)\|_H^2 \leq \|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)\|_H \eta(t)\|x(t) - y(t)\|_H,$$

for a.e. $t \in]0, T[$. By integrating, by (4.17c) and by the Gronwall type inequality,

$$\|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)\|_H \leq \int_0^t \eta(\tau)\|x(\tau) - y(\tau)\|_H \, d\tau, \text{ for each } t \in [0, T].$$

Thus, by a classical argument,

$$\|(\mathcal{T}^k x)(t) - (\mathcal{T}^k y)(t)\|_H \leq \frac{1}{k!}\|\eta\|_{L^1(0,T)}^k \|x - y\|_{C([0,T];H)}.$$

Hence $\mathcal{T}^k$ is a strict contraction for k large enough. By Banach's fixed point theorem $\mathcal{T}$ has a unique fixed point $u \in X$. Thus (4.4) has a unique solution.

Next we do not assume (H.6). Denote $M'_{u_0} = M_2(1 + \|u_0\|_H^2 + \phi(0, u_0))$, $M'_f = M_1\|f\|_{L^2(0,T;H)}^2$, $M'_0 = 5\|\eta\|_{L^1(0,T)} + 6M'_f + 12M'_{u_0}$, and $M'_1 = 48M_3\|\eta\|_{L^1(0,T)}^2 + 8\|\eta\|_{L^1(0,T)} + 80M_3M_f + 6M_4M_{u_0} + 6M_4$. Define, for $\alpha = 0, 1$,

$$Y_\alpha = \left\{ y \in C([0, T]; H) \,\middle|\, \|y\|_{C([0,T];H)}^2 + \int_0^T t^\alpha \psi(t, y(t))\, dt \leq M'_\alpha \right\}.$$

Lemma 4.7. Assume either the conditions of Theorem 4.3 ($\alpha = 0$) or those of Theorem 4.4 ($\alpha = 1$). In both cases, Y_α is nonempty, convex and closed in $C([0, T]; H)$, $\mathcal{T} : Y_\alpha \mapsto Y_\alpha$ is continuous and $\overline{\mathcal{T}Y_\alpha}$ is compact in $C([0, T]; H)$.

Proof. By Theorem 4.1 or 4.2, (4.1) has a solution $u \in C([0, T]; H)$. By (4.2) or by (4.3) and by (H.8), $u \in Y_\alpha$. By Fatou's lemma and by the lower semicontinuity of $\psi(\cdot)$, Y_α is closed. The convexity of Y_α is clear.

Assume the conditions of Theorem 4.3. Let $x \in Y_0$. Then $\mathcal{T}x \in Y_0$, since by (4.2) and by (H.8),

$$\|\mathcal{T}x\|_{C([0,T];H)}^2 + \int_0^T \psi(t, (\mathcal{T}x)(t))\, dt \leq M_1\|f - Cx\|_{L^2(0,T;H)}^2 + M'_{u_0} + \|\eta\|_{L^1(0,T)}$$
$$\leq \frac{2M_1}{\beta}\int_0^T \psi(t, x(t))\, dt + \frac{2M_1}{\beta}\|x\|_{C([0,T];H)}^2 + (1 + \frac{2M_1}{\beta})\|\eta\|_{L^1(0,T)} + 2M'_f + M'_{u_0} \leq M'_0.$$

Let $x, x_n \in Y_0$, $n = 1, 2 \dots$, be such that $x_n \to x$ in $C([0, T]; H)$, as $n \to \infty$. Denote $u_n = \mathcal{T}x_n$. By Theorem 4.1 there are v_n and w_n such that (u_n, v_n, w_n) is a solution of (4.17) with x_n instead of x. Moreover, by (4.2) and by (H.8),

$$\int_0^T \left(\|u'_n(\tau)\|_H^2 + \|v_n(\tau)\|_H^2 + \|w_n(\tau)\|_H^2\right) d\tau \leq M'_0. \tag{4.18}$$

Thus there are a subsequence and $u^*, v, w, c \in L^2(0,T;H)$ such that, as $n \to \infty$,

$$u_n' \to u^*,\ v_n \to v,\ w_n \to w,\ \text{and } Cx_n \to c \text{ weakly in } L^2(0,T;H).$$

By (H.7), there is $S \subset [0,T]$ of zero measure such that $\{u_n(t) \mid t \in [0,T] \setminus S, n = 1,2,\dots\}$ is relatively compact in H. By Ascoli's theorem [**D**, p. 143] and by the equicontinuity of $\{u_n\}$, there are a subsequence and $\tilde{u} \in C([0,T] \setminus S; H)$ such that $u_n \to \tilde{u}$ in $C([0,T] \setminus S; H)$, as $n \to \infty$. Thus $u_n \to \tilde{u}$ in $L^2(0,T;H)$, as $n \to \infty$. As in the proof of Theorem 4.1, $\tilde{u}' = u^*$, $v \in \partial\phi\tilde{u}$ and $w \in B\tilde{u}$. By (H.9), $c = C\tilde{u}$. We extend $\tilde{u}$ to a function from $C([0,T];H)$. By the uniqueness result of Theorem 4.1, $\tilde{u} = u = \mathcal{T}x$. Then,

$$\|\mathcal{T}x_n - \mathcal{T}x\|_{C([0,T];H)} = \|u_n - \tilde{u}\|_{C([0,T]\setminus S;H)} \to 0 \text{ as } n \to \infty.$$

This limit holds for the whole sequence, since otherwise there were a subsequence and another solution for (4.17). Hence $\mathcal{T} : Y_0 \mapsto Y_0$ is continuous.

Let $y_n \in \mathcal{T}Y_0$, $n = 1,2,\dots$ Then y_n satisfy (4.18). Again (y_n) has a subsequence converging in $C([0,T];H)$. Since $\mathcal{T}Y_0 \subset Y$ and Y is closed, then $\overline{\mathcal{T}Y_0}$ is sequentially compact. Hence it is compact [**KA**, I.5.1].

Assume the conditions of Theorem 4.4. Let $x \in Y_1$. Then by (4.3) and (H.8),

$$\|\mathcal{T}x\|^2_{L^\infty(0,T;H)} + \int_0^T t\psi(t,(\mathcal{T}x)(t))\,dt \le \frac{13}{12}M_3M_{Cx} + 4M_3M_f + M_4M_{u_0} +$$
$$+ M_4\|\eta\|_{L^1(0,T)} \le \frac{13M_3}{4\beta}\Big(\|x\|^2_{L^\infty(0,T;H)} + \int_0^T t\psi(t,x(t))\,dt\Big) + \frac{3M_1'}{16} \le M_1',$$

where also $ab \le \frac{1}{24}a^2 + 6b^2$, (H.10), the Cauchy-Bunyakovsky inequality and $(a+b+c)^2 = 2a^2 + \frac{5}{2}b^2 + 8c^2$ for $a,b,c \ge 0$, were used. Thus $\mathcal{T}x_1 \in Y_1$, i.e. $\mathcal{T}Y_1 \subset Y_1$.

Let $x_n, x \in Y_1$, $n = 1,2,\cdots$, with $x_n \to x$ in $C([0,T];H)$, as $n \to \infty$. Denote $u_n = \mathcal{T}x_n$. There are v_n and w_n satisfying (4.17) with Cx_n instead of Cx, and

$$\|u_n\|^2_{C([0,T];H)} + \operatorname{ess\,sup}\{t\phi_+(t,u_n(t)) \mid t \in [0,T]\} +$$
$$+ \int_0^T t\Big(\|u_n'(t)\|^2_H + \|v_n(t)\|^2_H + \|w_n(t)\|^2_H\Big)\,dt \le M_1'. \quad (4.19)$$

The weighted space $L^2(0,T;H,t)$ is a Hilbert space. By (H.7), there are $t' \in]0,T]$, $u^*, v, w, c \in L^2(0,T;H,t)$, $u \in L^2(0,T;H)$, and $\tilde{u} \in H$ such that, for a subsequence,

$$u_n' \to u^*,\ v_n \to v,\ w_n \to w,\ Cx_n \to c \text{ weakly in } L^2(0,T;H,t),$$
$$u_n \to u \text{ weakly in } L^2(0,T;H) \text{ and } u_n(t') \to \tilde{u} \text{ in } H,\ \text{as } n \to \infty.$$

Let $\delta > 0$. Again $u^* = u'$ a.e. on $[\delta, T]$ and thus we may redefine u on a set of zero measure such that $u \in H^1(\delta, T; H)$. Since δ arbitrary, $u \in C(]0,T]; H)$. Indeed,

$$u(t) = \tilde{u} + \int_{t'}^{t} u'(\tau)\, d\tau \text{ and } u_n(t) \to u(t) \text{ weakly in } H \text{ on }]0,T], \text{ as } n \to \infty. \tag{4.20}$$

By (H.7) there is $S \subset [\delta, T]$ of zero measure such that $\{u_n\}$ is an equicontinuous family of mappings from $[\delta, T] \setminus S$ to a compact metric space. By Ascoli's theorem [**D**, p. 143], there are a further subsequence (n_k) and $\hat{u}$ such that

$$\|u_{n_k} - \hat{u}\|_{C([\delta,T]\setminus S; H)} \to 0, \text{ as } k \to \infty. \tag{4.21}$$

However, by (4.20), $\hat{u}(t) = u(t)$, for each $t \in [\delta, T] \setminus S$. Thus (4.21) holds for the whole original subsequence. Since u and u_n are continuous on $[\delta, T]$,

$$\|u_n - u\|_{C([\delta,T];H)} = \|u_n - u\|_{C([\delta,T]\setminus S;H)} \to 0, \text{ as } n \to \infty. \tag{4.22}$$

By (4.17) and by the monotonicity of $\partial\phi(t, \cdot)$ and of $B(t)$,

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|u_n(t) - u_0\|_H^2 &= (u_n(t) - u_0, f(t) - C(t)x_n(t) - v_n(t) - w_n(t))_H \\ &\le (1 + \frac{1}{4\epsilon})\eta_1(t) + \|u_n(t) - u_0\|_H\big(\|C(t)x_n(t)\|_H + \eta_1(t)\big) + \epsilon t\|v_n(t) + w_n(t)\|_H^2, \end{aligned}$$

for a.e. $t \in]0, T[$ and for any $\epsilon > 0$; here $\eta_1, \eta_2 \in L^1(0,T)$ are independent on ϵ and on n. We integrate, use (4.19) and the Gronwall type inequality. Then, by (H.10) and by $x_n \in Y_1$, for each $t \in [0,T]$ and $\epsilon \in]0,1[$,

$$\begin{aligned} \|u_n(t) - u_0\|_H &\le \Big(4\epsilon M_1' + 2(1 + \frac{1}{\epsilon})\int_0^t \eta_1(\tau)\, d\tau\Big)^{\frac{1}{2}} + \int_0^t \eta_1(\tau)\, d\tau + \\ &+ \int_0^t \|C(\tau)x_n(\tau)\|_H\, d\tau \le 2\sqrt{\epsilon M_1'} + (1 + M_1')\epsilon + \epsilon^{-2}\int_0^t \eta_2(\tau)\, d\tau. \end{aligned} \tag{4.23}$$

Define $u(0) = u_0$ and let $\delta \in]0, T[$, $\epsilon \in]0,1[$. Then by (4.20),

$$\begin{aligned} \|u_n - u\|_{C([0,T];H)} &\le \sup_{0<t\le\delta} \|u_n(t) - u(t)\|_H + \sup_{\delta<t\le T} \|u_n(t) - u(t)\|_H \\ &\le \sup_{0<t\le\delta} \big(\|u_n(t) - u_0\|_H + \liminf_{m\to\infty} \|u_0 - u_m(t)\|_H\big) + \|u_n - u\|_{C([\delta,T];H)} \\ &\le \sup_{0<t\le\delta} \sup_{m=1,2,\ldots} 2\|u_m(t) - u_0\|_H + \|u_n - u\|_{C([\delta,T];H)}. \end{aligned}$$

Then, by (4.23) and by (4.22),

$$\limsup_{n\to\infty} \|u_n - u\|_{C([0,T];H)} \le 4\sqrt{\epsilon M_1'} + 2\epsilon(1+M_1') + 2\epsilon^{-2}\int_0^\delta \eta_2(\tau)\,d\tau$$
$$\to 4\sqrt{\epsilon M_1'} + 2\epsilon(1+M_1') \to 0, \text{ as } \delta\to 0,\ \epsilon\to 0, \text{ successively.}$$

Hence $u_n \to u$ in $C([0,T];H)$, as $n\to\infty$, and thus $u\in C([0,T];H)$.

By the demiclosedness result, by Lemma 4.1 and by (H.9),

$$v(t)\in\partial\phi(t,\cdot)u(t),\ w(t)\in B(t)u(t),\ c(t)=C(t)u(t),\ \text{for a.e. } t\in]\delta,T[$$

and for each $\delta\in]0,T[$. Thus u,v,w satisfy (4.17). Since the solution of (4.17) is unique in u, $u_n\to u=\mathcal{T}x$ in $C([0,T];H)$, as $n\to\infty$. Thus $\mathcal{T}$ is continuous.

Let $u_n\in\mathcal{T}Y_1$, $n=1,2,\ldots$ As above, there are a subsequence and $u\in C([0,T];H)$ such that $u_n\to u$ in $C([0,T];H)$, as $n\to\infty$. Since Y_1 is closed and $\mathcal{T}Y_1\subset Y_1$, then $\overline{\mathcal{T}Y_1}$ is sequentially compact. Thus $\overline{\mathcal{T}Y_1}$ is compact.

Lemma 4.7 is proved.

By Shauder's fixed point theorem [**KA**, XVI, 4.2], $\mathcal{T}: Y_\alpha\mapsto Y_\alpha$ has a fixed point $u\in Y_\alpha$, $\alpha=0,1$. Theorems 4.3 and 4.4 are proved.

5. The case of time-dependent G and K

Let $\delta,M,T>0$, $H=L^2(0,1)$, $V=H^1(0,1)$, and $\eta\in L^1(0,\max(1,T))$. We state on $g,k:[0,T]\times[0,1]\times\mathbf{R}\mapsto]-\infty,\infty]$ and $j:[0,T]\times\mathbf{R}^2\mapsto]-\infty,\infty]$, for each $t\in[0,T]$ and for a.e. $x\in]0,1[$:

(J_1) $g(t,\cdot,\cdot)$ and $k(t,\cdot,\cdot)$ are measurable with respect to the σ-field generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in $\mathbf{R}$.

(J_2) $g(t,x,\cdot)$ and $k(t,x,\cdot)$ are proper convex lower semicontinuous functions.

(J_3) $j(t,\cdot)$ is a proper convex lower semicontinuous function.

Denote $G(t,x)=\partial g(t,x,\cdot)$, $K(t,x)=\partial k(t,x,\cdot)$ and $\beta(t)=\partial j(t,\cdot)$.

(J_4) $(\tilde y_1-\tilde y_2)(y_1-y_2)\ge\delta(y_1-y_2)^2$, for each $(y_1,\tilde y_1),(y_2,\tilde y_2)\in G(t,x)$.

(J_5) There is $z:[0,T]\mapsto V$ such that $k(t,\cdot,z(t,\cdot))$, $g(t,\cdot,z_x(t,\cdot))\in L^1(0,1)$, $G(t,\cdot)^0z_x(t,\cdot)$, $K(t,\cdot)^0z(t,\cdot)\in H$, and $(z(t,0),z(t,1))\in D(\beta(t))$.

(J_6) For each $m>0$, there is $\eta_{m,t}\in H$ such that $|K(t,x)^0y|\le\eta_{m,t}(x)$, whenever $y\in[-m,m]$.

(J_7) Either $\beta(t)$ is bounded or $G(t,x)$ is bounded.

Remark 5.1. ([**BP**, p. 116]). Assume J_2 and, for a.e. $x\in]0,1[$ and each $t\in[0,T]$, $\operatorname{int}D(g(t,x,\cdot))\ne\emptyset\ne\operatorname{int}D(k(t,x,\cdot))$. Then J_1 is equivalent to J'_1.

(J'_1) For each $t\in[0,T]$ and $y\in\mathbf{R}$, $g(t,\cdot,y)$ and $k(t,\cdot,y)$ are measurable.

Define $\phi:[0,T]\times H\mapsto]-\infty,\infty]$,

$$\phi(t,y)=\begin{cases}\int_0^1 g(t,x,y'(x))\,dx+\int_0^1 k(t,y(x))\,dx+\\ +j(t,y(0),y(1)),\ \text{if } y\in H^1(0,1),\\ \infty, \qquad\qquad \text{otherwise,}\end{cases} \tag{5.1}$$

$$A(t) = \{(u, -w' + \hat{w}) \mid u, w \in V, \hat{w} \in H, w(x) \in G(t,x)u'(x),$$
$$\hat{w}(x) \in K(t,x)u(x), \text{ for a.e. } x \in]0,1[, (w(0), -w(1)) \in \beta(t)(u(0), u(1))\}. \quad (5.2)$$

Lemma 5.1. Assume J_1-J_5 and let $t \in [0,T]$. Then $\phi(t,\cdot)$ is a proper convex lower semicontinuous function. If, in addition, J_6 and J_7 are satisfied, then $\partial\phi(t,\cdot) = A(t)$ and $u_\lambda(t) = (I + \lambda A(t))^{-1}y$ satisfies, for some constant $M_\delta > 0$ and for any $y \in H$, $\lambda \in]0,1]$,

$$\|u_\lambda(t)\|_H^2 + \lambda\|u_\lambda(t)\|_V^2 \le M_\delta\Big(\|y\|_H^2 + \|z(t)\|_H^2 + \|K(t,\cdot)^0 z(t)\|_H^2 + $$
$$+ \lambda\|G(t,\cdot)^0 z_x(t,\cdot)\|_H^2 + \lambda\|\beta^0(t)(z(t,0), z(t,1))\|_{\mathbf{R}^2}^2\Big). \quad (5.3)$$

Let us state more hypotheses. Lemma 5.1 will be proved later.

(J_8) There is $h_0 \in]0,T[$ such that for each $\lambda \in]0,1]$, $h \in]0,h_0[$, $t \in [0,T]$, $y \in \mathbf{R}$, $\mathbf{y} \in \mathbf{R}^2$ and for a.e. $x \in [0,1]$,

$$(\tilde{y}_1 - \tilde{y}_2)(y_1 - y_2) \ge -\eta(t)h^2\Big(\eta(x) + y_1^2 + y_2^2 + g_+(t+h, y_1) + g_+(t, y_2)\Big) +$$
$$+\delta(y_1 - y_2)^2, \text{ whenever } (y_1, \tilde{y}_1) \in G(t+h, x), (y_2, \tilde{y}_2) \in G(t,x),$$
$$\|\beta_\lambda(t \pm h)\mathbf{y} - \beta_\lambda(t)\mathbf{y}\|_{\mathbf{R}^2}^2 \le \eta(t)h^2\Big(1 + \|\mathbf{y}\|_{\mathbf{R}^2}^2 + \|\beta_\lambda(t)\mathbf{y}\|_{\mathbf{R}^2} + j_{\lambda+}(t, \mathbf{y})\Big),$$
$$\sup\{|\xi| \mid \xi \in G(t,x)y\} \le M|y| + \sqrt{\eta(x)}.$$

(J_9) $\operatorname{ess\,sup}\{\|K^0(t,\cdot)z(t)\|_H + \|z(t)\|_H + |\phi(t, z(t))| + \|\beta^0(t)(z(t,0), z(t,1))\|_{\mathbf{R}^2} + \|G^0(t,\cdot)z_x(t,\cdot)\|_H \mid t \in [0,T]\} < \infty$.

(J_{10}) For each $h \in]0,h_0[$, $t \in]0, T-h[$, $y \in \mathbf{R}$, $\mathbf{y} \in \mathbf{R}^2$, $\lambda \in]0,1]$, and for a.e. $x \in]0,1[$,

$$\big|g(t+h,x,y) - g(t,x,y)\big| \le h\sqrt{\eta(t)}\Big(\eta(x) + y^2 + |G(t,x)^0 y| + g_+(t,x,y)\Big),$$
$$\big|j_\lambda(t+h,\mathbf{y}) - j_\lambda(t,\mathbf{y})\big| \le \sqrt{\eta(t)}\Big(1 + \|\mathbf{y}\|_{\mathbf{R}^2}^2 + \|\beta_\lambda(t)\mathbf{y}\|_{\mathbf{R}^2} + j_{\lambda+}(t,\mathbf{y})\Big).$$

Lemma 5.2. Assume J_1-J_9 and let $y \in H$, $\lambda \in]0,1]$, $g, j \ge 0$, and $k \equiv 0$. Then $u_\lambda \in H^1(0,T;V)$ and there is a constant $M^* > 0$, satisfying for a.e. $t \in]0,T[$,

$$\|u_\lambda'(t)\|_H^2 + \lambda\|u_\lambda'(t)\|_V^2 \le \lambda M^*\Big(1 + \|u_\lambda(t)\|_V^2 + \phi(t, u_\lambda(t)) +$$
$$+ \|\partial\phi_\lambda(t,\cdot)y\|_H + \limsup_{h\to 0+}\int_0^1 g(t+h, x, u_{\lambda x}(x,t))\,dx\Big). \quad (5.4)$$

Proof. (Hint). We estimate $\int_0^{T-h}\|u_\lambda(t+h) - u_\lambda(t)\|_V^2\,dt$ using $u_\lambda + \lambda A u_\lambda \ni y$ and the approximate problems with $\beta_\mu(t)$ instead of $\beta(t)$ and their limits as $\mu \to 0+$, cf. the end of the proof of Lemma 5.1.

Lemma 5.3. Assume the conditions of Lemma 5.2 and J_{10}. Let $u_0 \in H$ and $f \in L^1(0,T;H)$ be such that $M_f + M_{u_0} < \infty$ and define $B(t) \equiv 0$. Then all the conditions of Theorem 4.2 are satisfied.

Proof. (Hint). For (H.3) we estimate $\int_0^{T-h} |\phi_\lambda(t+h,y) - \phi_\lambda(t,y)|^2 \, dt$.

By Theorem 4.2, (4.3), J_4 and by J_9, there are thus $u \in C([0,T]; L^2(0,1))$, differentiable a.e. on $]0,T[$, and measurable $w : [0,T] \mapsto H^1(0,1)$ satisfying (1.1) with $K(t,x) \equiv 0$ and

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_0^1 t\big(u_x(t,x)^2 + w(t,x)^2\big)\, dx + \int_0^T \int_0^1 t\big(u_t(t,x)^2 + w_x(t,x)^2\big)\, dx\, dt < \infty.$$

Proof of Lemma 5.1. Since $G(t,x)$ and $K(t,x)$ are subdifferentials,

$$\begin{aligned} g(t,x,y) &\ge g(t,x,z_x(t,x)) + G(t,x)^0 z_x(t,x)(y - z_x(t,x)), \\ k(t,x,y) &\ge k(t,x,z(t,x)) + K(t,x)^0 z(t,x)(y - z(t,x)). \end{aligned}$$

Thus g and k are normal convex integrands. Hence the integrals in (5.1) are well-defined, taking values from $]-\infty,\infty]$. The convexity of $\phi(t,\cdot)$ is clear. In order to prove the lower semicontinuity, let $\lambda > 0$ and consider the level sets $S_\lambda = \{y \in H \mid \phi(t,y) \le \lambda\}$. Let $y_n \in S_\lambda$ and $y \in H$, $n = 1,2,\ldots$, be such that $y_n \to y$ in H, as $n \to \infty$. By J_4, J_5 and by the definition for subdifferential we obtain, for a.e. $x \in]0,1[$,

$$g(t,x,y_n'(x)) \ge \frac{\delta}{8} y_n'(x)^2 + g(t,x,z_x(t,x)) - 2G(t,x)^0 z_x(t,x) - \delta z(t,x)^2. \tag{5.6}$$

By (5.1) and by J_5, (y_n) is bounded in V. So, for a subsequence, $y_n \to y$ weakly in V, as $n \to \infty$. By Mazur's lemma we can form convex combinations z_n of y_n's such that $z_n \to y$ strongly in V. Thus $z_n \in S_\lambda$ and for a subsequence,

$$z_n(x) \to y(x), \text{ for each } x \in [0,1], \text{ and } z_n'(x) \to y'(x), \text{ for a.e. } x \in]0,1[.$$

By Fatou's lemma we obtain now (cf. [**BP**, p. 117]) that $y \in S_\lambda$, since

$$\begin{aligned} \lambda \ge \liminf_{n\to\infty} \int_0^1 g(t,x,z_n'(x))\, dx + \liminf_{n\to\infty} \int_0^1 k(t,x,z_n(x))\, dx + \\ + \liminf_{n\to\infty} j(t, z_n(0), z_n(1)) \ge \phi(t,y). \end{aligned}$$

Hence $\phi(t,\cdot)$ is lower semicontinuous. Thus $\partial\phi(t,\cdot)$ exists; evidently it contains $A(t)$. So $A(t) \subset H \times H$ is monotone. Hence it is maximal, if $R(I + A(t)) = H$.

Lemma 5.4. (Cf. [**M1**, p. 250]). Let $\gamma \subset \mathbf{R}^2 \times \mathbf{R}^2$ be maximal monotone and

$$D(F) = \{w \in H^2(0,1) \mid (w'(0), w'(1)) \in \gamma(w(0), -w(1))\},\ (Fw)(x) = -w''(x).$$

Then F is maximal monotone in H.
Proof. Clearly, F is monotone. Let us prove that $R(I+F) = H$, i.e.

$$-\,w''(x) + w(x) = y(x), \quad \text{for a.e. } x \in]0,1[, \tag{5.7a}$$
$$(w'(0), w'(1)) \in \gamma(w(0), -w(1)). \tag{5.7b}$$

has a solution $u \in H^2(0,1)$, for each $y \in H$. The general solution of (5.7a) is $w(x) = c_1\mathrm{e}^x + c_2\mathrm{e}^{-x} + w_1(x)$ where w_1 is some solution of (5.7a). Thus (5.7b) is satisfied if $\gamma\mathbf{y} + \hat{\gamma}\mathbf{y} \ni \mathbf{y}_1$, where $\mathbf{y}_1 \in \mathbf{R}^2$ depends only on w_1 and

$$\hat{\gamma} = \frac{1}{\mathrm{e}^2 - 1}\begin{pmatrix} 1+\mathrm{e}^2 & 2\mathrm{e} \\ 2\mathrm{e} & 1+\mathrm{e}^2 \end{pmatrix},\ \mathbf{y} = \begin{pmatrix} 1 & 1 \\ -\mathrm{e} & -\mathrm{e}^{-1} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} w_1(0) \\ -w_1(1) \end{pmatrix}.$$

Since $\hat{\gamma}$ is continuous, monotone and coercive, $\gamma + \hat{\gamma}$ is surjective [**Ba**, p. 48], and thus $\gamma\mathbf{y} + \hat{\gamma}\mathbf{y} \ni \mathbf{y}_1$ has a solution $\mathbf{y} \in \mathbf{R}^2$. Lemma 5.4 is proved.

Let $y \in H$ and $y_n \in C_0^\infty(]0,1[)$, $n = 1, 2, \ldots$, be such that $y_n \to y$ in H, as $n \to \infty$. Consider the problems (5.8);

$$-w_n''(x) + G(t,x)^{-1}w_n(x) + \frac{1}{n}w_n(x) = y_n'(x), \text{ for a.e. } x \in]0,1[, \tag{5.8a}$$
$$(w_n'(0), w_n'(1)) \in \beta(t)^{-1}(w_n(0), -w_n(1)). \tag{5.8b}$$

Choosing in Lemma 5.4 $\gamma = \beta(t)^{-1}$ and denoting by $\hat{F}$ the realization of $G(t,x)^{-1} + 1/n$ to H, we see that (5.8) is equivalent to $Fw_n + \hat{F}w_n = y_n'$. Since $F + \hat{F}$ is surjective [**Ba**, p. 48], (5.8) has a solution $w_n \in H^2(0,1)$. Define

$$u_n(x) = w_n'(0) + \int_0^x G(t,\sigma)^{-1}w_n(\sigma)\,d\sigma,\ v_n(x) = u_n(x) + \frac{1}{n}\int_0^x w(\sigma)\,d\sigma, \tag{5.9a}$$

for each $x \in [0,1]$. Then $u_n, v_n \in H^1(0,1)$ and

$$v_n(x) - w_n'(x) = y_n(x), \text{ for each } x \in [0,1], \tag{5.9b}$$
$$w_n(x) \in G(t,x)u_n'(x), \text{ for a.e. } x \in]0,1[, \tag{5.9c}$$
$$(w_n(0), -w_n(1)) \in \beta(t)\big(v_n(0), v_n(1)\big). \tag{5.9d}$$

We multiply (5.9b) by $v_n - z(t)$ and integrate over $[0,1]$. By J_2-J_5 we obtain

$$\|u_n\|_{H^1(0,1)}^2 + \|v_n\|_{H^1(0,1)}^2 + \|w_n'\|_H^2 + \frac{1}{n}\|w_n\|_H^2 \le M^* < \infty. \tag{5.10}$$

If $\beta(t)$ is bounded, then $(w_n(0), -w_n(1))$ is bounded. Assume that $G(t,x)$, $x \in [0,1]$, are bounded and $(w_n(0), -w_n(1))$ is unbounded. Then, by (5.10), $w_n(x)$ is unbounded, for each $x \in [0,1]$. By Fatou's lemma,

$$\liminf_{n\to\infty} \int_0^1 u_n'(x)^2\,dx \geq \int_0^1 \liminf_{n\to\infty} \Big(G(t,x)^{-1} w_n(x)\Big)^2\,dx = \infty,$$

which contradicts (5.10). Hence in any case $(w_n(0), -w_n(1))$ is bounded. Thus we have a subsequence and $u, w \in H^1(0,1)$ such that, as $n \to \infty$,

$$\begin{gathered} v_n \to u,\ w_n \to w \text{ in } H, \text{ and } u_n' \to u',\ \ w_n' \to w', \text{ weakly in } H, \\ v_n(0) \to u(0),\ \ v_n(1) \to u(1),\ w_n(0) \to w(0),\ w_n(1) \to w(1). \end{gathered}$$

Since the realization of $G(t,x)$ in H and $\beta(t)$ are maximal monotone,

$$u(x) - w'(x) = y(x), \text{ for a.e. } x \in]0,1[, \tag{5.11a}$$
$$w(x) \in G(t,x)u'(x), \text{ for a.e. } x \in]0,1[, \tag{5.11b}$$
$$(w(0), -w(1)) = \beta(t)(u(0), u(1)). \tag{5.11c}$$

Hence $\hat{A}(t) = A(t)$, given by (5.2) with $K(t,x) \equiv 0$, is maximal monotone.

Let $m > 0$ and $K(t)$, $K^m(t)$ be the maximal monotone realizations to H of $K(t,x)$ and of $K^m(t,x)$, respectively; $K^m(t,x)$ is maximal monotone, bounded by $\eta_{m,t}(x)$ and equals to $K(t,x)$ on $[-m,m]$. Then $D(K^m(t)) = H$. By [**Br**, p. 36], $\hat{A}(t) + K^m(t)$ is maximal monotone. Let $y \in H$. Then there is $(u_m, v_m) \in \hat{A}(t)$ such that $u_m + v_m + K^m(t)u_m \ni y$. We multiply this by $u_m - z(t)$ and obtain u_m to be bounded in $H^1(0,1)$, independently on m. We choose m to be so big that $K^m(t)u_m$ equals $K(t)u_m$. Thus $R(I + A(t)) = H$.

From $u_\lambda(t) + \lambda A(t)u_\lambda(t) \ni y$ we obtain (5.3). Lemma 5.1 is proved.

Remark 5.2. Assume J_1-J_2 and J_4-J_7 and let $\beta(t) \subset \mathbf{R}^2 \times \mathbf{R}^2$ be maximal monotone. Then $A(t)$, given by (5.2), is maximal monotone.

Indeed, $A(t)$ is clearly monotone. For $R(I + A(t)) = H$ in the proof of Lemma 5.1 we did not need $\beta(t)$ to be a subdifferential.

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