

A Class of Degenerate Multivalued Second-Order Boundary Value Problems*

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The main purpose of this paper is to investigate existence and uniqueness for 1D, double nonlinear and multivalued, degenerate, second-order boundary value problems of the form

$$0 \in -(p(r)G(u'(r)))' + q(r)H(u(r)), \quad 0 < r < 1, \quad (1.1)$$

$$0 \in (p(r)G(u'(r)))|_{r=0^+}, \quad C \in p(1)G(u'(1)). \quad (1.2)$$

Our assumptions are sufficiently general to cover a wide class of applications. We show by some appropriate examples that the assumptions are quite natural and essential for the validity of our main result stated in Theorem 2.1. A variational interpretation of the above equations is also given. © 1999 Academic Press

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1. INTRODUCTION

The aim of this paper is to investigate the above 1D, double nonlinear and multivalued, possibly degenerate, second-order boundary value problem (BVP).

The precise meaning of this BVP will be explained later, but it is a natural extension of the classical problem in which we have equalities instead of inclusions.

Let us now introduce:

Assumption (A1). $G: D(G) \subset R \rightarrow R$ is a maximal monotone mapping (possibly multivalued), G is strictly monotone (i.e., strictly increasing), and the pair $(0, 0)$ belongs to the graph of G ;

Assumption (A2). $p \in C(0, 1]$, $p(r) > 0$ for all $r \in (0, 1]$; $q \in L^1(0, 1)$, $q(r) > 0$ for a.e. $r \in (0, 1)$; for every Lipschitz continuous and nondecreasing function $z: [0, 1] \rightarrow R_+ = [0, \infty)$, the application

$$r \rightarrow \frac{1}{p(r)} \int_0^r q(s) z(s) ds$$

is also nondecreasing in $(0, 1]$, and finally we have

$$\lim_{r \rightarrow 0^+} \frac{1}{p(r)} \int_0^r q(s) ds = 0; \quad (1.3)$$

(Remark that the limit (1.3) does exist, because of the very previous hypothesis in which we take $z(r) = 1$, but we impose this limit to be zero.)

Assumption (A3). C is a real constant such that $C/p(1) \in R(G)$, where $R(G)$ denotes the range of G ;

Assumption (A4). $H: D(H) \subset R \rightarrow R$ is a maximal monotone mapping (possibly multivalued) and $(0, 0)$ belongs to the graph of H ; if $C > 0$ there exists $\gamma \in D(H)$, $\gamma > \beta := G^{-1}(C/p(1))$, such that

$$\frac{CC_1}{p(1)} < \sup H(\gamma - \beta), \quad (1.4)$$

and, respectively, if $C < 0$ there exists $\gamma \in D(H)$, $\gamma < \beta$, such that

$$\frac{CC_1}{p(1)} > \inf H(\gamma - \beta), \quad (1.4')$$

where

$$C_1 := p(1) \left/ \int_0^1 q(s) ds \right.$$

Remarks. 1. By Assumption (A2) there follows

$$C_1 = \inf_{0 < r \leq 1} \frac{p(r)}{\int_0^r q(s) ds}.$$

2. If $\gamma - \beta \in \text{Int } D(H)$ it follows by the well known Rockafellar's theorem [9, Chap. I] that $H(\gamma - \beta)$ is a bounded set. In fact, it is a bounded closed interval of real numbers (possibly a singleton), because H is a maximal monotone mapping. Therefore $\sup H(\gamma - \beta)$ and $\inf H(\gamma - \beta)$ are finite numbers. It is also possible that $\beta = 0$ and γ may be the right or left end of the interval $D(H)$ and, in this case, the right-hand side of (1.4) (respectively, (1.4')) is $+\infty$ (respectively, $-\infty$).

3. As G and H are assumed to be nonlinear and multivalued, it is natural to say that BVP (i.e., the problem (1.1), (1.2)) is *double nonlinear and multivalued*.

4. Our assumptions allow the function p to vanish at $r = 0$ (more precisely, $p(0^+) = 0$) or, even to have a singularity at $r = 0$ (for example, $p(r) = r^a$, $q(r) = r^b$ with $a, b \in R$, $b + 1 > \max(0, a)$). That is why we call our BVP *possibly degenerate* (e.g., according to the terminology of S. Mikhlin (Ed.) [8, Chap. 7] for linear elliptic partial differential equations).

In all that follows we shall suppose that Assumptions (A1)–(A4) hold if not otherwise stated. In order to clarify the meaning of our BVP, let us give some notions of solutions and discuss them by means of some appropriate examples.

DEFINITION 1.1. By a solution of BVP we mean a function $u \in C^1[0, 1]$ such that

$$u(r) \in D(H), u'(r) \in D(G) \quad \text{for all } r \in [0, 1], \quad (1.5)$$

$$u'(1) = \beta := G^{-1}(C/p(1)), \quad (1.6)$$

and there exists a function $v \in AC[0, 1]$ satisfying

$$v(r) \in p(r)G(u'(r)), \quad \text{for all } r \in (0, 1], \quad (1.7)$$

$$v'(r) \in q(r)H(u(r)), \quad \text{a.e. } r \in (0, 1), \quad (1.8)$$

$$v(0) = 0. \quad (1.9)$$

We have denoted, by $C^1[0, 1]$, the space of continuously differentiable functions: $[0, 1] \rightarrow R$ and, by $AC[0, 1]$, the space of absolutely continuous functions: $[0, 1] \rightarrow R$.

We may also consider the following concept of a solution to BVP:

DEFINITION 1.2. $u \in C^1[0, 1]$ is a solution of BVP if u satisfies Definition 1.1 except for (1.6) which is replaced by

$$v(1) = C. \quad (1.10)$$

Obviously, if u is a solution in the sense of the last definition then it is also solution in the sense of Definition 1.1. In general the converse is not true, as Examples 1.1 and 1.2 below show. Hence, the second concept is stronger than the first. As one can observe immediately, in the second case we have uniqueness, at least up to an additive constant, while in the first case that may not happen (see, also, Examples 1.1 and 1.2). That is a consequence of the fact that G is multivalued. Of course, if $GG^{-1}(C/p(1)) = C/p(1)$ then the two notions of solution are identical. This is the case for any C satisfying (A3) if G is in addition a single valued mapping.

Example 1.1. Take $p(r) = q(r) = r$; $C \geq 0$; $H(\xi) = \xi$, $\xi \in R$, and let G be defined by

$$G(\xi) = \begin{cases} \xi & \text{if } \xi < 0 \\ [0, 1] & \text{if } \xi = 0 \\ \xi + 1 & \text{if } \xi > 0. \end{cases}$$

It is easy to see that Assumptions (A1)–(A4) are all satisfied.

Let u be a solution of this BVP in the sense of Definition 1.1. Then u satisfies

$$\begin{aligned} u'(r) &= G^{-1}\left(r^{-1} \int_0^r su(s) ds\right) \\ &= G^{-1}\left(r^{-1} \int_0^r s \left(u(0) + \int_0^s u'(t) dt\right) ds\right). \end{aligned} \quad (1.11)$$

From this equation we can see that if $u(0) > 0$ then $u' \geq 0$ in $[0, 1]$ and hence (see again (1.11)) u' is nondecreasing in $[0, 1]$. Similarly, if $u(0) < 0$ then u' is nonpositive and nonincreasing in $[0, 1]$. If $u(0) = 0$ the Gronwall lemma applied to (1.11) shows that u is the null function. On the other hand (1.11) implies that $u'(0) = 0$. If $C \in [0, 1]$ then (1.6) reads $u'(1) = 0$ and hence u' is identically zero, because of the monotonicity. It is then

easy to see that, for $C \in [0, 1]$, the constant functions $u(r) = C_1$, $C_1 \in [0, 2]$ are solutions in the sense of Definition 1.1. Hence, we have existence, without uniqueness. On the other hand, for each $C \in [0, 1]$ our BVP admits the unique solution $u(r) = 2C$, in the sense of Definition 1.2.

For $C > 1$, the conditions (1.6) and (1.10) coincide and therefore the two concepts of solution are identical. In this case, Theorem 2.1 below guarantees existence and uniqueness.

Example 1.2. Take $p(r) = q(r) = 1$; $C \geq 0$; $H(\xi) = \xi$, $\xi \in R$, and $G: R \rightarrow R$, defined by

$$G(\xi) = \begin{cases} \xi & \text{for } \xi < 1 \\ [1, 2] & \text{for } \xi = 1 \\ \xi + 1 & \text{for } \xi > 1 \end{cases}$$

First of all, it is easy to prove the uniqueness of the solution of BVP in the sense of Definition 1.2. (We shall reconsider this point in the general framework of our assumptions.)

Now, let $u \in C^1[0, 1]$ be a solution of BVP in the sense of Definition 1.1. Then, we can write the identity

$$v(r)u(r) = \int_0^r \{v(s)u'(s) + u(s)^2\} ds.$$

This implies the equality

$$\{r \in (0, 1]; u(r) = 0\} = \{r \in (0, 1]; u'(r) = 0\} \quad (1.12)$$

and this set is either the empty set or an interval of the form $(0, \delta]$. If $C = 0$, u is clearly the null function. Now, suppose that $C > 0$. Then $u'(1) > 0$ and hence, according to the above remark concerning the form of the set (1.12), $u' \geq 0$ in $[0, 1]$. From the obvious equation

$$u'(r) = G^{-1}\left(\int_0^r u(s) ds\right) \quad (1.13)$$

we can deduce that $u(0) \geq 0$ and hence $u \geq 0$ in $[0, 1]$. Looking again at (1.13) we then deduce that u' is nondecreasing in $[0, 1]$. Now, if $0 < C < 1$ the set $U = \{r \in [0, 1]; u'(r) < 1\}$ coincides to $[0, 1]$. Therefore, in this case u satisfies the problem

$$u'' = u \quad \text{in } [0, 1]; \quad u'(0) = 0, \quad u'(1) = C,$$

which has a unique solution

$$u(r) = \frac{C}{e - e^{-1}}(e^r + e^{-r}), \quad 0 \leq r \leq 1. \quad (1.14)$$

In fact, in this case the two concepts of solution coincide.

Now, for every $C \in [1, 2]$ we have the same boundary value conditions

$$u'(0) = 0, \quad u'(1) = 1.$$

As u' is nondecreasing, the interval $[0, 1]$ can be decomposed into two subintervals:

$$U = \{r \in [0, 1]; u'(r) < 1\} = [0, r_0),$$

$$V = \{r \in [0, 1]; u'(r) = 1\} = [r_0, 1],$$

where $r_0 \in (0, 1]$. An elementary computation shows us that u is given by the formula

$$u(r) = \begin{cases} \frac{e^r + e^{-r}}{e^{r_0} - e^{-r_0}}, & 0 \leq r \leq r_0 \\ r - r_0 + \frac{e^{r_0} + e^{-r_0}}{e^{r_0} - e^{-r_0}}, & r_0 \leq r \leq 1, \end{cases} \quad (1.15)$$

for all $r_0 \in (0, 1]$, verifying the inequality

$$(1 - r_0)\coth r_0 + (1 - r_0)^2/2 \leq 1. \quad (1.16)$$

So, we may conclude that for every $C \in [1, 2]$ BVP has an infinite number of solutions in the sense of Definition 1.1, the same solutions for every $C \in [1, 2]$.

Now, we ask ourselves, what about the solutions in the sense of Definition 1.2, for $C \in [1, 2]$?

First, for $C = 1$ the (unique) solution in the sense of Definition 1.2 is given by (1.14), with $r_0 = 1$. Let us now take $C \in (1, 2]$ and denote by u_C the corresponding solution in the sense of Definition 1.2, assuming that it does exist. Then clearly there exists a number $r_0 \in (0, 1)$ such that u_C coincides to u given by (1.15) with that r_0 . An easy computation, involving all the conditions of Definition 1.2, shows that r_0 should necessarily satisfy the condition

$$(1 - r_0)\coth r_0 + (1 - r_0)^2/2 = C - 1. \quad (1.17)$$

But (1.17) has a unique solution and hence, for every $C \in [1, 2]$, BVP has a unique solution in the sense of Definition 1.2.

Finally, for $C > 2$ the two notions of solution coincide again, because (1.6) and (1.10) are identical: $u'(1) = C - 1$. Therefore, in this case there exists a unique solution, given by Theorem 2.1 below. In fact, we can precisely indicate the solution in this case:

$$u(r) = \frac{C - 1}{e - e^{-1}}(e^r - e^{-r}), \quad 0 \leq r \leq 1. \quad (1.18)$$

Remark that we have the same solution u_C for $C = 1$ and for $C = 2$. On the other hand, we can see from (1.17) that r_0 depends continuously on $C \in [1, 2]$. Therefore, taking into account (1.14), (1.15), and (1.18), we can deduce that u_C depends continuously on C .

We recommend to the reader to discuss, also, the same example but with $p(r) = q(r) = r$. This is a multivalued and degenerate problem and similar aspects can be observed. Of course, in this case the boundary value condition at $r = 0$ is automatically satisfied.

2. EXISTENCE AND UNIQUENESS

Particular cases of BVP have been studied by Corduneanu and Moroşanu [2, 3], Moroşanu [10–12], Moroşanu and Corduneanu [13], and Moroşanu and Zofotă [14, 15]. The main progress of the present work is the fact that both G and H are allowed to be multivalued, i.e., our BVP is double multivalued. This generalization is nontrivial and covers nice and important applications.

The main result of this section is

THEOREM 2.1. *If Assumptions (A1)–(A4) hold, then BVP has a solution in the sense of Definition 1.2, which is unique up to an additive constant. If in addition H is strictly increasing too, then the solution in the sense of Definition 1.2 is unique.*

Before proving this result let us discuss our assumptions, using several adequate examples. First, we remark that the strict monotonicity of G is essential. Otherwise, it is possible that our BVP has no solution, even if all other assumptions of Theorem 2.1 hold. Here is an example.

Example 2.1. Let $G: R \rightarrow R$ be the single valued (but not strictly monotone) function defined by

$$G(\xi) = \begin{cases} \xi & \text{for } \xi < 1 \\ 1 & \text{for } 1 \leq \xi \leq 2 \\ \xi - 1 & \text{for } \xi > 2. \end{cases}$$

Consider the following BVP (which satisfies (A1)–(A4) except for the strict monotonicity of G)

$$\begin{aligned} (rG(u'))' &= ru, & 0 < r < 1 \\ G(u'(1)) &= 2. \end{aligned} \tag{2.1}$$

Remark that in this case the first condition of (1.2) is superfluous whereas the second one coincides to $u'(1) = 3$, that is, the two concepts of solution are identical. Let us suppose that (2.1) has a solution u . From the obvious equation

$$G(u'(r)) = (1/r) \int_0^r su(s) ds$$

we can see that $u'(0) = 0$. On the other hand, in the open set

$$U = \{r \in (0, 1); 1 < u'(r) < 2\}$$

the function u satisfies the equation $1 = ru(r)$ and this implies that in fact U is empty. But this contradicts the Darboux property of u' . Therefore (2.1) has no solution.

Example 2.2. Take $p(r) = q(r) = 1$; $H(\xi) = \xi$, $\xi \in R$;

$$G(\xi) = \begin{cases} 0 & \text{for } \xi \leq 1 \\ \xi - 1 & \text{for } \xi > 1, \end{cases}$$

and $C \geq 0$.

It is easy to see that the solution of this BVP in the sense of Definition 1.2 is unique for any $C \geq 0$. For $C = 0$ this is the null function. We can observe that in this case, Definition 1.1 (see (1.6)) should be changed. (In fact, even in the previous example (1.6) does not make sense in that form if $C = 1$.) If $C > 0$ then (1.6) and (1.10) are identical: $u'(1) = C + 1$. On the other hand, we have

$$\{r \in [0, 1]; u'(r) < 1\} \subset \{r \in [0, 1]; u(r) = 0\}$$

and consequently, as u' has the Darboux property and $u'(1) > 1$, we necessarily have $u' \geq 1$ in $[0, 1]$. Therefore, for $C > 0$ BVP is equivalent to

$$\begin{aligned} u'' &= u, & 0 < r < 1 \\ u'(0) &= 1, & u'(1) = C + 1 \\ u' &\geq 1 \text{ in } [0, 1]. \end{aligned} \tag{2.2}$$

But (2.2) has the (unique) solution

$$u(r) = C_1(e^r + e^{-r}) + e^r, \quad C_1 = (C + 1 - e)/(e - e^{-1}).$$

This example shows, however, that there are particular situations where the existence is possible without strict monotonicity for G . That is also possible in the degenerate case, as the following examples show.

Example 2.3. $p(r) = q(r) = r$; $H(\xi) = \xi$, $\xi \in R$;

$$G(\xi) = \begin{cases} \xi & \text{for } \xi \leq 1 \\ 1 & \text{for } \xi > 1, \end{cases}$$

and $C \leq 1$. Clearly, the set $\{r \in [0, 1]; u'(r) > 1\}$ is empty and hence BVP is equivalent to

$$\begin{aligned} (ru')' &= ru & 0 < r < 1 \\ G(u'(1)) &= C \\ u' &\leq 1 \text{ in } [0, 1]. \end{aligned} \tag{2.2'}$$

For $C < 1$ the boundary condition is equivalent with $u'(1) = C$ and we already know that (2.2') has a unique solution [11]. For $C = 1$ we can use only Definition 1.2 and the solution in this sense is unique. To prove its existence, we take in (2.2') the boundary value condition $u'(1) = 1$ and the resulting problem has a unique solution (see also [11]).

Example 2.4. Take the same elements as in Example 2.3 except for G which is assumed to be the multivalued Heaviside function

$$G(\xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ [0, 1] & \text{if } \xi = 0 \\ 1 & \text{if } \xi > 0, \end{cases}$$

and $0 \leq C \leq 1$. We leave to the reader to verify that for each $C \in [0, 1]$ BVP has the unique solution $u(r) = 2C$, in the sense of Definition 1.2. For every $C \in [0, 1]$ the constant functions $u(r) = C_1$, $0 \leq C_1 \leq 2$ are solutions in the sense of Definition 1.1.

We also leave to the reader to consider the same example but with G replaced by the multivalued sign function.

The last three examples show that Assumption (A1) is not even minimal. Certainly, the most relevant is Example 2.1 that shows that in general we cannot expect existence without the strict monotony for G .

The same Darboux property indicates to us that G and H should be assumed to be maximal monotone mappings, i.e., their graphs are continuous lines in R^2 . The next example will clarify this point.

Example 2.5. $p(r) = q(r) = 1$; $H(\xi) = \xi$, $\xi \in R$;

$$G(\xi) = \begin{cases} \xi & \text{if } \xi \leq 1 \\ \xi + 1 & \text{if } \xi > 1, \end{cases}$$

and $C = 3$. Assuming that BVP admits a solution $u \in C^1 [0, 1]$, there follows that $u'(0) = 0$ and $u'(1) = 2$. Hence the range of u' is an interval I which includes $[0, 2]$. But $G(I)$ is not an interval and therefore u cannot satisfy the equation

$$G(u'(r)) = \int_0^r u(s) ds \quad 0 \leq r \leq 1.$$

This situation will not appear again if G is replaced by the multivalued extension

$$\tilde{G}(\xi) = \begin{cases} G(\xi) & \text{if } \xi \in R - \{1\} \\ [1, 2] & \text{if } \xi = 1, \end{cases}$$

which is a maximal monotone mapping. Similar arguments show us that H must also be maximal monotone. In fact, as we shall see, it is enough to assume that G and H are restrictions of maximal monotone operators, such that their graphs are continuous lines.

As regards Assumption (A2), this is technical and perhaps could be weakened. But it covers a wide class of applications.

In what follows, we shall construct two examples which show that condition (1.4) (or (1.4')) is not only essential but even minimal for the existence.

Example 2.6. Take $p(r) = q(r) = 1$; $C = 1$; $G, H: R \rightarrow R$, $G(\xi) = \xi^{2k+1}$, where k is a natural number, and

$$H(\xi) = \begin{cases} \xi & \text{if } \xi \leq a \\ a & \text{if } \xi > a, \end{cases}$$

where a is a positive number.

As G is strictly monotone, the two notions of solution coincide. If $a > 1$ Ineq. (1.4) of Assumption (A4) is satisfied and the existence for BVP is assured by Theorem 2.1 below.

Now, we consider the case $0 < a < 1$, for which (1.4) is not valid anymore, and suppose that BVP has a solution $u \in C^1[0, 1]$. We multiply (1.1) by $u(r)$ and integrate on $[0, r]$

$$u(r) \cdot u'(r)^{2k+1} = \int_0^r \{u'(s)^{2k+2} + u(s)H(u(s))\} ds. \quad (2.3)$$

From (2.3) we can see that

$$\{r \in (0, 1]; u(r) = 0\} = \{r \in (0, 1]; u'(r) = 0\} \quad (2.4)$$

and this set is either an empty set or an interval of the form $(0, \delta]$. As $u'(1) = 1$ it follows that $u' \geq 0$ in $[0, 1]$, so u is nondecreasing in $[0, 1]$. By (2.3) and (2.4) it follows that $u \geq 0$ in $[0, 1]$ and this implies that u' is nonincreasing, because (see (1.1))

$$u'(r) = \left(\int_0^r H(u(s)) ds \right)^{1/(2k+1)}.$$

In particular

$$0 = u'(0) \leq u'(r) \leq 1, \quad \text{for } 0 \leq r \leq 1. \quad (2.5)$$

On the other hand, multiplying Eq. (1.1) by u' and then integrating on $[0, r]$ we get

$$\frac{2k+1}{2k+2} u'(r)^{2k+2} = h(u(r)) + \text{Const.}, \quad (2.6)$$

where

$$h(\xi) := \begin{cases} \xi^2/2 & \text{if } \xi < a \\ a(2\xi - a)/2 & \text{if } \xi \geq a. \end{cases}$$

From (2.5) and the Mean Value theorem it follows that there exists a point $\alpha \in (0, 1)$ such that

$$\frac{2k+1}{2k+2} = h(u(1)) - h(u(0)) = H(u(\alpha))u'(\alpha).$$

Therefore (see also (2.5))

$$\frac{2k+1}{2k+2} \leq a,$$

but this inequality is impossible for k big enough. Consequently, for such k BVP has no solution!

The limit case $a = 1$, for which we have equality in (1.4), remains open.

Example 2.7. We propose to the reader to take the same elements as in the previous example except for H which is replaced by the (strictly

increasing) function

$$H(\xi) = a \cdot \arctan \xi, \quad \xi \in R,$$

where $a \in (0, 2/\pi)$. Clearly (1.4) is not satisfied and repeating, step by step, the reasoning used in the previous example we can show that BVP has no solution for large k . For $a > 2/\pi$ inequality (1.4) holds and Theorem 2.1 says that BVP has a unique solution. In the limit case $a = 2/\pi$ the inequality (1.4) is still not satisfied. In this case BVP has a unique solution (see Remark 2.3 below). However, this is a limit case.

The above two examples show very clearly that, even in the case in which BVP is nondegenerate, the contribution of the nonlinearity H is very important for the existence (by (1.4) or (1.4')) it should be "big enough").

Let us finish this long but necessary discussion by presenting a very simple example (in fact, a counterexample) which shows that if in Theorem 2.1 H is not strictly increasing then the solution of BVP may not be unique (of course, it is however unique up to an additive constant).

Example 2.8. $p(r) = q(r) = 1$; $C = 1$; $G(\xi) = \xi$, $\xi \in R$, and

$$H(\xi) = \begin{cases} \xi & \text{if } \xi < 1 \\ 1 & \text{if } 1 \leq \xi \leq 2 \\ \xi - 1 & \text{if } \xi > 2. \end{cases}$$

As (A1)–(A4) are all satisfied, the existence is assured by Theorem 2.1. Moreover, it is easily seen that all the functions

$$u(r) = r^2/2 + C_1, \quad 1 \leq C_1 \leq 3/2 \quad (2.7)$$

are solutions of the corresponding BVP,

$$u'' = H(u), \quad u'(0) = 0, \quad u'(1) = 1.$$

In fact, there are no other solutions of BVP. Indeed, by Theorem 2.1 below we have uniqueness up to an additive constant and, on the other hand, u given by (2.7) with $C_1 < 1$ or $C_1 > 3/2$ cannot be a solution of Eq. (1.1).

Now, we are going to the

Proof of Theorem 2.1. We mention that some ideas come from previous work [2, 3, 14, 15] but, for completeness, we present the full proof, with several improvements of the previous arguments. The proof is divided into several steps.

Step 1. Uniqueness. Let $u_1, u_2 \in C^1[0, 1]$ be two solutions of BVP in the sense of Definition 1.2 and let $v_1, v_2 \in AC[0, 1]$ be the corresponding selections given by that definition. Using (1.7)–(1.9) we can easily obtain that

$$0 = \int_0^1 \{(v_1 - v_2)(u'_1 - u'_2) + q(u_1 - u_2)(w_1 - w_2)\}, \quad (2.8)$$

where

$$w_i(r) \in H(u_i(r)) \quad \text{for a.e. } r \in (0, 1), \quad i = 1, 2$$

such that

$$v'_i(r) = q(r)w_i(r), \quad \text{for a.e. } r \in (0, 1), \quad i = 1, 2.$$

As H is nondecreasing and G is strictly increasing, (2.8) yields $u'_1 = u'_2$. If in addition H is strictly increasing too, then (2.8) implies that $u_1 = u_2$.

Step 2. Reducing to the case $C > 0$ and $\beta > 0$.

Clearly, for $C = 0$ the null function is a solution of BVP. In what follows we shall discuss only the case $C > 0$, because for $C < 0$ we can use similar arguments. Furthermore, we shall assume that $\beta > 0$. The case $\beta = 0$ is a little bit different and will be solved below.

Step 3. Associating an auxiliary BVP.

We fix a $C > 0$ satisfying (A3). Assuming that $\beta > 0$, we define $\tilde{G}, \tilde{H}: R \rightarrow R$ as follows

$$\tilde{G}(\xi) = \begin{cases} \xi & \text{if } \xi \leq 0 \\ G(\xi) & \text{if } 0 < \xi < \beta \\ [G^0(\xi), C/p(1)] & \text{if } \xi = \beta \\ \xi - \beta + C/p(1) & \text{if } \xi \geq \beta, \end{cases}$$

$$\tilde{H}(\xi) = \begin{cases} \xi & \text{if } \xi \leq 0 \\ H(\xi) & \text{if } 0 < \xi < \gamma \\ \xi - \gamma + H^0(\gamma) & \text{if } \xi \geq \gamma, \end{cases}$$

where β and γ are the constants appearing in (A4) and

$$G^0(\beta) := \inf G(\beta) \quad H^0(\gamma) = \inf H(\gamma).$$

As $G(\beta)$ and $H(\gamma)$ are closed intervals, we have $G^0(\beta) \in G(\beta)$ and $H^0(\gamma) \in H(\gamma)$. Clearly, \tilde{G} and \tilde{H} are maximal monotone mappings. (For information concerning monotone operator theory, we refer the reader to

[1, 9].) By replacing G, H in BVP with \tilde{G}, \tilde{H} we obtain a problem, which will be called (BVP'):

$$0 \in -(p\tilde{G}(u'))' + q\tilde{H}(u), \quad 0 < r < 1 \quad (1.1')$$

$$0 \in p\tilde{G}(u')|_{r=0^+}, \quad C \in p(1)\tilde{G}(u'(1)). \quad (1.2')$$

Step 4. Solving a Cauchy problem associated to a regularized equation.

As \tilde{G} is strictly monotone and maximal monotone, the operator $\tilde{F} = \tilde{G}^{-1}$ is single valued and maximal monotone. For the time being we assume, in addition to (A1)–(A4), that

$$\begin{aligned} \tilde{F} \text{ is Lipschitz continuous and } \tilde{H} \text{ is single valued} \\ \text{and Lipschitz continuous too.} \end{aligned} \quad (2.9)$$

We are going to solve the problem

$$u'(r) = \tilde{F}\left(\frac{1}{p(r)} \int_0^r q(s)\tilde{H}(u(s)) ds\right), \quad (2.10)$$

$$u'(1) = \beta. \quad (2.11)$$

First, we consider the Cauchy problem made up by Eq. (2.10) and the initial value condition

$$u(0) = u_0. \quad (2.12)$$

Denoting $y := u'$ this Cauchy problem can be written as the integral equation

$$y(r) = \tilde{F}\left(\frac{1}{p(r)} \int_0^r q(s)\tilde{H}\left(u_0 + \int_0^s y(t) dt\right) ds\right). \quad (2.13)$$

Due to (A2), Eq. (2.13) makes sense at $r = 0$. We now state

LEMMA 2.1. *If Assumptions (A2) and (2.9) hold, then for every $u_0 \in R$, Eq. (2.13) has a unique solution $y = y(r, u_0) \in C[0, 1]$.*

Proof of Lemma 2.1. One applies the Banach Fixed Point Principle to the operator $T: C[0, 1] \rightarrow C[0, 1]$, $(Ty)(r) :=$ the right-hand side of Eq. (2.13). It suffices to observe that T is a contraction with respect to the norm

$$\|y\|_{C[0,1]} := \sup\{e^{-2Lt}|y(t)|; 0 \leq t \leq 1\},$$

if L is a positive and sufficiently large constant.

Q.E.D.

Step 5. Proving that for regular \tilde{G} and \tilde{H} (BVP') has a solution in the sense of Definition 1.1.

Suppose that (A1)–(A4) and (2.9) hold. We recall that $y(r, u_0)$ denotes the solution of the Cauchy problem ((2.10) and (2.12)). In what follows, the equality

$$\{y(1, u_0); u_0 \geq 0\} = [0, \infty). \quad (2.14)$$

will be proved. In order to do this, we need some properties of $y(r, u_0)$. First, it is clear that

$$y(r, 0) = 0, \quad \text{for } 0 \leq r \leq 1. \quad (2.15)$$

Now, it is easily seen that

$$u_0 > 0 \quad \text{implies} \quad y(r, u_0) \geq 0, \quad \text{for } 0 \leq r \leq 1 \quad (2.16)$$

and

$$y(0, u_0) = 0, \quad \text{for every } u_0 \in R. \quad (2.17)$$

Indeed, if $u_0 > 0$ then $u_0 + \int_0^s y(t, u_0) dt \geq 0$ in some interval $0 \leq s \leq \delta$ and hence, by (2.13), $y(r, u_0) \geq 0$ for $0 \leq r \leq \delta$. In fact this interval can be extended to the right up to an interval $[0, \delta_{\max}]$ in which $y(r, u_0) \geq 0$. Moreover, δ_{\max} is in fact 1 and so (2.16) is proved. As regards (2.17), this is a consequence of (1.3).

Now, using again (2.13) and Gronwall's lemma we can derive the Lipschitz property

$$|y(r, u_0) - y(r, \bar{u}_0)| \leq K|u_0 - \bar{u}_0|, \quad (2.18)$$

for all $u_0, \bar{u}_0 \in R$ and $0 \leq r \leq 1$, where K is a positive constant. On the other hand, since

$$y(1, u_0) \geq \tilde{F} \left(\frac{1}{p(1)} \tilde{H}(u_0) \int_0^1 q(s) ds \right),$$

we have

$$y(1, u_0) \rightarrow \infty \quad \text{as } u_0 \rightarrow \infty. \quad (2.19)$$

From (2.15), (2.16), (2.18), and (2.19), it follows (2.14) as a consequence of the Darboux property. Clearly (2.14) shows that there exists a $\bar{u}_0 \geq 0$ such that $y(1, \bar{u}_0) = \beta$ and hence the function

$$u(r) = \bar{u}_0 + \int_0^r y(s, \bar{u}_0) ds$$

is a solution of problem (2.10), (2.11). In fact, $\bar{u}_0 > 0$ because $y(1, 0) = 0$ (see (2.15)).

Step 6. Eliminating the assumption (2.9).

Replace the functions \tilde{F} , \tilde{G} by their Yosida approximations \tilde{F}_λ , \tilde{H}_λ , $\lambda > 0$ [9, p. 20]:

$$\tilde{F}_\lambda := \frac{1}{\lambda}(I - J_\lambda) = \tilde{F}J_\lambda, \quad J_\lambda := (I + \lambda\tilde{F})^{-1}.$$

It is well known that \tilde{F}_λ , \tilde{H}_λ , $\lambda > 0$, are Lipschitz continuous. Therefore, according to Lemma 2.1 and Step 5, for each $\lambda > 0$, there exists a solution y_λ of (2.13), with \tilde{F}_λ and \tilde{H}_λ instead of \tilde{F} and \tilde{H} , satisfying $y_\lambda(1) = \beta$. In fact, for each $\lambda > 0$, there exists a $u_{0\lambda} > 0$ such that $u_\lambda \in C^1[0, 1]$ defined by

$$u_\lambda(r) = u_{0\lambda} + \int_0^r y_\lambda(s) ds \quad (2.20)$$

satisfies the problem

$$u'_\lambda(r) = \tilde{F}_\lambda \left(\frac{1}{p(r)} \int_0^r q(s) \tilde{H}_\lambda(u_\lambda(s)) ds \right) \quad (2.21)$$

$$u'_\lambda(1) = \beta. \quad (2.22)$$

As $y_\lambda \geq 0$ (see (2.16)) it follows by (2.20) that u_λ is nondecreasing. Now, Assumption (A2) comes again into play, showing that u'_λ is also nondecreasing (see (2.21)). In particular, we have that

$$0 = u'_\lambda(0) \leq u'_\lambda(r) \leq \beta, \quad \text{for all } \lambda > 0, 0 \leq r \leq 1. \quad (2.23)$$

Now, we are going to prove that, for some $\lambda_0 > 0$ fixed, the set

$$\{u_\lambda; 0 < \lambda \leq \lambda_0\} \text{ is bounded in } C[0, 1]. \quad (2.24)$$

To this purpose it suffices to show that the set $\{u_{0\lambda}; 0 < \lambda \leq \lambda_0\}$ is bounded (cf. (2.20) and (2.23)). Indeed, we have that

$$\beta = u'_\lambda(1) \geq \tilde{F}_\lambda \left(\frac{1}{p(1)} \tilde{H}_\lambda(u_{0\lambda}) \int_0^1 q(s) ds \right) \geq 0. \quad (2.25)$$

On the other hand, a simple computation shows us that, for ξ large enough and $0 < \lambda \leq \lambda_0$,

$$\tilde{H}_\lambda(\xi) = \frac{1}{1 + \lambda} (\xi - \gamma + H^0(\gamma))$$

and

$$\tilde{F}_\lambda(\xi) = \frac{1}{1 + \lambda}(\xi + \beta - C/p(1)).$$

This remark and (2.25) imply the boundedness of the set $\{u_{0\lambda}; 0 < \lambda \leq \lambda_0\}$ as claimed. From (2.23) and (2.24) it follows by virtue of the Arzela–Ascoli Criterion that there exists a function $u \in C[0, 1]$ such that, on a subsequence,

$$u_\lambda \rightarrow u \quad \text{in } C[0, 1] \quad \text{as } \lambda \rightarrow 0^+. \quad (2.26)$$

Since the resolvent of \tilde{H} , say J_λ^H , is nonexpansive and $J_\lambda^H(0) = 0$, we have for $0 < \lambda \leq \lambda_0$

$$|J_\lambda^H u_\lambda(r)| \leq |u_\lambda(r)| \leq \text{Const.} \quad (2.27)$$

Obviously, \tilde{H} is bounded on bounded sets and this implies, by virtue of (2.27), that

$$|\tilde{H}_\lambda(u_\lambda(r))| \leq C_1, \quad (2.28)$$

for $0 < \lambda \leq \lambda_0$, $0 \leq r \leq 1$. Therefore

$$\begin{aligned} |J_\lambda^H u_\lambda(r) - u(r)| &\leq |J_\lambda^H u_\lambda(r) - u_\lambda(r)| + |u_\lambda(r) - u(r)| \\ &\leq C_1 \lambda + |u_\lambda(r) - u(r)|, \end{aligned}$$

which implies (see (2.26)) that

$$J_\lambda^H u_\lambda \rightarrow u \quad \text{in } C[0, 1] \quad \text{as } \lambda \rightarrow 0^+, \quad (2.29)$$

on the same subsequence as in (2.26). Using (2.28) and (2.29) and the fact that \tilde{H} is closed (as a multivalued mapping) we can see that there exists a function $w \in L^\infty(0, 1)$ such that

$$\tilde{H}_\lambda(u_\lambda(r)) \rightarrow w(r) \in \tilde{H}(u(r)) \quad \text{as } \lambda \rightarrow 0^+, \quad \text{for } 0 \leq r \leq 1. \quad (2.30)$$

Consequently,

$$\begin{aligned} \frac{1}{p(r)} \int_0^r q(s) \tilde{H}_\lambda(u_\lambda(s)) ds &\rightarrow \frac{1}{p(r)} \int_0^r q(s) w(s) ds \quad \text{as } \lambda \rightarrow 0^+, \\ 0 &\leq r \leq 1. \end{aligned} \quad (2.31)$$

In fact, (2.30) and (2.31) hold, also, with respect to the weak-star topology of $L^\infty(0, 1)$. By a similar reasoning for \tilde{F}_λ we can pass to the limit in (2.21)

and (2.22) to find that u belongs to $C^1[0, 1]$ and satisfies

$$u'(r) = \tilde{F}\left(\frac{1}{p(r)} \int_0^r q(s)w(s) ds\right), \quad 0 \leq r \leq 1, \quad (2.32)$$

$$u'(1) = \beta, \quad (2.33)$$

i.e., u is a solution of (BVP') in the sense of Definition 1.1.

Step 7. Existence for BVP.

Consider a sequence $C_n > C$, with $C_n \rightarrow C$, and denote by $(BVP)_n$ our BVP with \tilde{G} instead of G , \tilde{H} instead of H , and C_n instead of C . We put

$$\beta_n := \tilde{G}^{-1}(C_n/p(1)) = \beta + \frac{C_n - C}{p(1)}.$$

Taking into account the above reasoning, we can say that for each n problem $(BVP)_n$ has a solution in the sense of Definition 1.1, say u_n . More precisely, for each n there exists $w_n \in L^\infty(0, 1)$ such that

$$w_n(r) \in \tilde{H}(u_n(r)), \quad 0 \leq r \leq 1, \quad (2.34)$$

$$u'_n(r) = \tilde{F}\left(\frac{1}{p(r)} \int_0^r q(s)w_n(s) ds\right), \quad 0 \leq r \leq 1, \quad (2.35)$$

$$u'_n(1) = \beta_n. \quad (2.36)$$

Moreover, u_n are solutions for $(BVP)_n$ in the sense of Definition 1.2. Using again the above arguments we can write that

$$0 \leq u'_n(r) \leq \beta_n, \quad 0 \leq r \leq 1, \quad (2.37)$$

$$0 \leq u_n(0) \leq u_n(r), \quad 0 \leq r \leq 1. \quad (2.38)$$

We are now going to prove that

$$u_n(0) \leq \gamma - \beta \quad \text{for } n \text{ sufficiently large.} \quad (2.39)$$

Indeed, otherwise we would have

$$\begin{aligned} \frac{C_n C_1}{p(1)} &= C_1 \tilde{G}(u'_n(1)) = \frac{C_1}{p(1)} \int_0^1 q(s) \tilde{H}(u_n(s)) ds \\ &\geq \tilde{H}(u_n(0)) \geq \sup \tilde{H}(\gamma - \beta) = \sup H(\gamma - \beta), \end{aligned}$$

and this contradicts (1.4) for n large enough. Now, by (2.37)–(2.39) we find that

$$0 \leq u_n(r) = u_n(0) + \int_0^r u_n'(s) ds \leq \gamma + \beta_n - \beta, \quad \text{for } 0 \leq r \leq 1. \quad (2.40)$$

From (2.37) and (2.40) we deduce, by virtue of the Arzela–Ascoli Criterion, that, on a subsequence,

$$u_n \rightarrow u \quad \text{in } C[0, 1]. \quad (2.41)$$

We shall prove that u is a solution of BVP in the sense of Definition 1.2. First, we can pass to the limit in (2.35). Indeed, there exists a function $w \in L^\infty(0, 1)$, with $w(s) \in \tilde{H}(u(s))$ a.e. in $(0, 1)$, such that

$$u'(r) = \tilde{F} \left(\frac{1}{p(r)} \int_0^r q(s)w(s) ds \right), \quad 0 \leq r \leq 1. \quad (2.42)$$

By (2.42) we can see that in fact $u \in C^1[0, 1]$. Moreover, according to (2.37) and (2.40), $u'(r) \in [0, \beta]$, $u(r) \in [0, \gamma]$ for all $r \in [0, 1]$, and hence we can put in (2.42) F, H instead of \tilde{F}, \tilde{H} . On the other hand, let us denote

$$v_n(r) = \int_0^r q(s)w_n(s) ds, \quad v(r) = \int_0^r q(s)w(s) ds.$$

Obviously, v_n are the functions associated with u_n in Definition 1.2. It is easy to see that

$$v_n \rightarrow v \quad \text{in } C[0, 1].$$

In particular, we have

$$v(1) = \lim v_n(1) = \lim C_n = C$$

and hence v satisfies Definition 1.2.

Step 8. Solving the case $C > 0$ and $\beta = 0$.

In this case we define \tilde{G} as

$$\tilde{G}(\xi) = \begin{cases} \xi & \text{if } \xi < 0 \\ [0, C/p(1)] & \text{if } \xi = 0 \\ \xi + C/p(1) & \text{if } \xi > 0. \end{cases}$$

If $\gamma \in \text{Int } D(H)$ we define \tilde{H} as above. If γ is such that $D(H) \cap R_+ = [0, \gamma]$ then we take $C_2 \in H(\gamma)$ such that $CC_1/p(1) < C_2$ and define \tilde{H} as

$$\tilde{H} = \begin{cases} \xi & \text{if } \xi \leq 0 \\ H(\xi) & \text{if } 0 < \xi < \gamma \\ [H^0(\gamma), C_2] & \text{if } \xi = \gamma \\ \xi + C_2 - \gamma & \text{if } \xi > \gamma. \end{cases}$$

With these slight modifications, the proof of the existence can be done as before. In fact, in this case the solution is a constant function. The proof of Theorem 2.1 is now complete.

Remark 2.1. If $GG^{-1}(C/p(1)) = C/p(1)$ then we can take in Step 7 of the above proof $C_n = C$, because in this case the solution of (BVP'), in the sense of Definition 1.1, is also a solution in the sense of Definition 1.2. The rest of the proof is the same.

Remark 2.2. If u is the solution given by Theorem 2.1, then necessarily $u'(0) = 0$.

Remark 2.3. An inspection of Step 7 shows that if H is also strictly increasing then we can put \leq in (1.4) if $C > 0$ (and, respectively, \geq in (1.4') if $C < 0$).

Remark 2.4. Looking again at the proof of Theorem 2.1 we can observe that it is sufficient to know the mappings G and H in the intervals $[0, \beta]$, $[0, \gamma]$ if $C > 0$ and, respectively, $[\beta, 0]$, $[\gamma, 0]$ if $C < 0$.

Remark 2.5. Theorem 2.1 says that in the framework of (A1)–(A4) there exists at least one solution in the sense of Definition 1.2 and so Definition 1.1 seems to be superfluous. However, beyond this framework, different situations may appear. For example, it is possible to have existence only in the sense of Definition 1.1. Here is an example.

Example 2.9. Take $p(r) = q(r) = r$; $C > 0$; $H(\xi) = \xi$, $\xi \in R$; and $G: D(G) = (\infty, 1] \rightarrow R$,

$$G(\xi) = \begin{cases} 0 & \text{if } \xi < 1 \\ [0, \infty) & \text{if } \xi = 1. \end{cases}$$

Clearly, Assumptions (A1)–(A4) are all satisfied except for the strict monotonicity of G . For any $C > 0$ condition (1.6) becomes $u'(1) = 1$. On the other hand, if u is a solution of BVP in the sense of Definition 1.1 then

$$\{r \in [0, 1]; u'(r) < 1\} \subset \{r \in [0, 1]; u(r) = 0\}$$

and hence, by the Darboux property, $u' = 1$, $0 \leq r \leq 1$. It is then easy to see that the functions

$$u(r) = r + C_1, \quad C_1 \geq 0$$

are all solutions of BVP in the sense of Definition 1.1. Now, we look for the solutions of BVP in the sense of Definition 1.2. The existence of such solutions is equivalent to the existence of some functions $v \in AC[0, 1]$ such that

$$v(r) \geq 0 \quad \text{for } 0 < r \leq 1, \quad (2.43)$$

$$v'(r) = r(r + C_1) \quad \text{a.e. } r \in (0, 1), \quad (2.44)$$

$$v(0) = 0, \quad v(1) = C. \quad (2.45)$$

Clearly, for $0 < C < 1/3$, the system (2.43)–(2.45) has no solution, hence BVP has no solution in the sense of Definition 1.2. For each $C \geq 1/3$ our BVP has a unique solution in the sense of Definition 1.2: $u(r) = r + 2(C - 1/3)$.

3. THE VARIATIONAL INTERPRETATION OF BVP

Suppose again that Assumptions (A1)–(A4) hold.

It is well known that any maximal monotone operator from R into R is the subdifferential of some proper, convex, lower semicontinuous function, which is uniquely determined up to an additive constant. So, $G = \partial g$ and $H = \partial h$, where $g, h: R \rightarrow (-\infty, +\infty]$ are both proper, convex, and lower semicontinuous. More precisely, we know that $D(G)$ and $D(H)$ are intervals and that g, h can be defined as [9, Chap. I]

$$g(\xi) = \begin{cases} \int_0^\xi G^0(t) dt & \text{for } \xi \in \text{Cl}D(G) \\ +\infty & \text{otherwise,} \end{cases} \quad (3.1)$$

$$h(\xi) = \begin{cases} \int_0^\xi H^0(t) dt & \text{for } \xi \in \text{Cl}D(H) \\ +\infty & \text{otherwise.} \end{cases} \quad (3.2)$$

Of course, $g + \text{Const.}$ and $h + \text{Const.}$ are also good functions for the same purpose. Furthermore, g is strictly convex because G is a strictly monotone mapping. In fact, we have the following simple result

PROPOSITION 3.1. *Let $j: R \rightarrow (-\infty, +\infty]$ be a proper convex function, such that its effective domain $D(j)$ is not a singleton. Then, j is strictly convex if and only if ∂j is strictly monotone.*

Now, let us define the functional $\Psi: W^{1,1}(0, 1) \rightarrow (-\infty, +\infty]$, by

$$\Psi(v) = \int_0^1 \{p(r)g(v'(r)) + q(r)h(v(r))\} dr - Cv(1). \quad (3.3)$$

It is only a simple exercise (involving the definition of subdifferential) to see that the solution of BVP in the sense of Definition 1.2, given by Theorem 2.1 above, is a minimizer of the (convex) functional Ψ . Therefore, a solution of BVP in the sense of Definition 1.2 is a *variational solution*, while a solution in the sense of Definition 1.1 is not necessarily a variational one (in fact it is a minimizer of Ψ given by (3.3) but possibly with another constant instead of C , which belongs to the interval $p(1)GG^{-1}(C/p(1))$). This interpretation seems to clarify the meaning of the two notions of solutions.

A similar interpretation can be done for the end $r = 0$. In fact, the solution in the sense of Definition 1.2 is a variational solution with respect to both ends $r = 0$ and $r = 1$, as it appears as a minimizer of the functional Ψ . On the other hand, as seen above (see Remark 2.2), the solution of BVP, in any of the two senses, satisfies $u'(0) = 0$. In general this is not equivalent to condition (1.9). Here is an example in this sense.

Example 3.1 [14]. Consider the equation

$$(r^{-1}u'(r))' = u(r), \quad 0 < r < 1. \quad (3.4)$$

According to Theorem 2.1, Eq. (3.4) with the boundary value conditions

$$\lim_{r \rightarrow 0^+} r^{-1}u'(r) = 0, \quad u'(1) = C \quad (3.5)$$

has a unique solution $u \in C^1[0, 1]$. Now, let us associate to Eq. (3.4) the boundary value conditions

$$u'(0) = 0, \quad u'(1) = C. \quad (3.6)$$

The general solution of Eq. (3.4) is given by

$$u(r) = r \left[c_1 I_{2/3}(2r^{3/2}/3) + c_2 I_{-2/3}(2r^{3/2}/3) \right], \quad (3.7)$$

where I_m represents the modified Bessel function of the first kind and of order m (see, e.g. [7, p. 301]), while c_1, c_2 are real constants. One remarks that u given by (3.7) satisfies $u'(0) = 0$ for any constants c_1, c_2 . Therefore, problem (3.4), (3.6) has an infinite number of solutions.

Remark 3.1. In fact, we may consider, in the above example, instead of Eq. (3.4) the more general equation

$$(r^{-1}u'(r))' = r^b u(r), \quad (3.8)$$

where $b > -1$. Denote $a = 2/(b + 3)$. The reader can easily see that by means of the substitutions

$$x = ar^{1/a}, \quad w = r^{-1}u$$

Eq. (3.8) can be written as the modified Bessel equation

$$x^2(d^2w/dx^2) + x(dw/dx) - (x^2 + a^2)w = 0.$$

4. FINAL COMMENTS

In this paper we have concentrated our attention on the problem of existence and uniqueness. We intend to continue our study to cover at least the following topics.

4.1. *Dependence on the data*

In general Theorem 2.1 does not guarantee the uniqueness of the solution. However, using again the technique from the proof of Theorem 2.1, we can easily obtain the following result of upper semicontinuity with respect to the parameter C :

If (A1)–(A4) hold and $C_n \rightarrow C$, then there exist u_n solutions in the sense of Definition 1.2 for BVP with C_n instead of C , such that $u_n \rightarrow u$ in $C^1[0, 1]$, at least on a subsequence, where u is a solution of BVP in the sense of Definition 1.2.

It is expected that the same technique may be applied to prove a result of continuity with respect to p , q , G , H , and C . Also, in the case of uniqueness, some results of differentiability and sensitivity of the solutions with respect to some parameters are expected.

4.2. *Applications*

The nondegenerate case of our BVP is a general model for a wide class of applications. A nice application of the degenerate case comes from the capillarity problem in circular tubes [4, 5, pp. 262–263, 6, pp. 289–293]. This model has also been considered in [2, 12, 13]. Other applications for BVP, including the multivalued case, are also possible.

4.3. *Variational Approach*

It is almost certain that the variational approach, as used in [11], also works for this more general case. This approach may allow us to consider more general problems, including the multidimensional or even infinite dimensional case. In addition, the variational approach could offer us efficient numerical algorithms.

All the above topics and perhaps other related subjects will be discussed in a forthcoming paper.

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