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A contraction proximal point algorithm with two monotone operators

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ABSTRACT

It is a known fact that the method of alternating projections introduced long ago by von Neumann fails to converge strongly for two arbitrary nonempty, closed and convex subsets of a real Hilbert space. In this paper, a new iterative process for finding common zeros of two maximal monotone operators is introduced and strong convergence results associated with it are proved. If the two operators are subdifferentials of indicator functions, this new algorithm coincides with the old method of alternating projections. Several other important algorithms, such as the contraction proximal point algorithm, occur as special cases of our algorithm. Hence our main results generalize and unify many results that occur in the literature.

1. Introduction

Consider the following convex feasibility problem:

$$\text{find an } x \in H \quad \text{such that } x \in K_1 \cap K_2, \tag{1}$$

where K_1 and K_2 are nonempty, closed and convex subsets of a real Hilbert space H with nonempty intersection. In his 1933 paper, von Neumann showed that problem (1) can be solved by means of an iterative process. Indeed, if K_1 and K_2 are closed vector subspaces of H, von Neumann showed that any sequence (x_n) generated from the method of alternating projections

$$H \ni x_0 \mapsto x_1 = P_{K_1}x_0 \mapsto x_2 = P_{K_2}x_1 \mapsto x_3 = P_{K_1}x_2 \mapsto x_4 = P_{K_2}x_3 \mapsto \cdots,$$

converges strongly to a solution of problem (1) that is closest to the starting point x_0. The reader interested in the proof of this classical result is referred to, for example, [1,2] and the references therein. For the case when K_1 and K_2 are two arbitrary nonempty, closed and convex subsets in H with nonempty intersection, it was Bregman [3] who first showed that the sequence (x_n) generated from the method of alternating projections converges weakly to a point in $K_1 \cap K_2$. Note that in this case strong convergence fails in general, as illustrated by Hundal [4], see also [5]. In order to enforce strong convergence

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of the method of alternating resolvents, the current authors proposed several modifications of this method [6]. The most general one was given in [7], see also [8]. Such a method generates a sequence (x_n) according to the rule

$$x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n) \quad \text{for } n = 0, 1, \ldots, \tag{2}$$

$$x_{2n} = J^B_{\mu_n}(\lambda_n u + (1-\lambda_n)x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \ldots, \tag{3}$$

for some given $u, x_0 \in H$, where (e_n) and (e'_n) are sequences of computational errors, A and B are maximal monotone operators, and $\alpha_n, \lambda_n \in (0, 1)$ and $\beta_n, \mu_n \in (0, \infty)$. Here $J^A_\beta := (I + \beta A)^{-1}$, $\beta > 0$ (the resolvent operator of A). It was shown in [6–8] that under appropriate assumptions on the sequences of real numbers (α_n), (λ_n), (β_n) and (μ_n), and the sequences of computational errors (e_n) and (e'_n), the sequence generated from the method of alternating resolvents (2), (3) converges strongly to a point in $A^{-1}(0) \cap B^{-1}(0) =: F$ which is nearest to the point u. The method (2), (3) is in fact an extension of the method given in [9]. In addition to finding a point in F, the method proposed in [9] is capable of finding fixed points of the composition mapping $J^A_\mu J^B_\mu$, where μ is a positive real number. We observe that the method of alternating resolvents given above is defined via the regularized proximal point algorithm. Since the proximal iterates can also be generated from the contraction proximal point algorithm (CPPA),

$$x_{n+1} = \alpha_n u + \delta_n x_n + \gamma_n J^A_{\beta_n} x_n + e_n \quad \text{for } n = 0, 1, \ldots$$

(where $\alpha_n, \delta_n, \gamma_n \in (0, 1)$ with $\alpha_n + \delta_n + \gamma_n = 1$), which also produces sequences that converge strongly for a single maximal monotone operator, we shall introduce and investigate convergence properties of sequences generated from the CPPA involving two maximal monotone operators A and B. (Note that for each $n \geq 0$ and $u \in H$ fixed, the $(n+1)$th iterate defines a contraction, hence the name CPPA). More precisely, for $\alpha_n, \delta_n, \gamma_n \in (0, 1)$ with $\alpha_n + \delta_n + \gamma_n = 1$ and $\lambda_n, \rho_n, \sigma_n \in (0, 1)$ with $\lambda_n + \rho_n + \sigma_n = 1$, we introduce the following algorithm:

$$x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n} x_{2n} + e_n \quad \text{for } n = 0, 1, \ldots, \tag{4}$$

$$x_{2n} = \lambda_n u + \rho_n x_{2n-1} + \sigma_n J^B_{\mu_n} x_{2n-1} + e'_n \quad \text{for } n = 1, 2, \ldots, \tag{5}$$

and will prove under minimal assumptions on the sequences of parameters defining (x_n) that the sequence (x_n) converges strongly to a point in F that is nearest to u. Algorithm (4), (5) contains as special cases the inexact proximal point algorithm introduced independently by Kamimura and Takahashi [10] and Xu [11] as well as the generalized contraction proximal point algorithm which was introduced by Yao and Noor [12]. Therefore, the results of this paper extend and unify many results such as [12, Theorem 3.3], [13, Theorem 1] and [14, Theorems 2–6]. It is worth mentioning that [9, Theorem 3.3] addresses the weak convergence of the sequence generated by the method of alternating resolvents (to a point in F), whereas our present paper addresses the issue of strong convergence of the sequence given by the above modified algorithm (see (4), (5)).

2. Some preliminaries

Throughout this paper, H will be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We recall that a map $T : H \to H$ is called nonexpansive if for every $x, y \in H$ we have $\|Tx - Ty\| \leq \|x - y\|$. The map T is called firmly nonexpansive if for every $x, y \in H$ we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2 .$$

It is easy to see that firmly nonexpansive mappings are nonexpansive. For more information on firmly nonexpansive mappings, we refer the reader to the excellent book by Goebel and Reich [15]. An operator $A : D(A) \subset H \to 2^H$ is said to be monotone if for every pair of points (x, y), (x', y') in the graph $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$ of A, we have $\langle x - x', y - y' \rangle \geq 0$. In other words, an operator is monotone if its graph is a monotone subset of the product space $H \times H$. An operator A is called maximal monotone if in addition to being monotone, its graph is not properly contained in the graph of any other monotone operator. Note that if A is maximal monotone, then so is its inverse A^{-1}. Given a maximal monotone operator A, one can define a single-valued and firmly nonexpansive mapping $J^A_\beta := (I + \beta A)^{-1}$ (where I is the identity operator), for every $\beta > 0$. This kind of operator is called the resolvent of A. It is known that the Yosida approximation of A, an operator defined by $A_\beta := \beta^{-1}(I - J^A_\beta)$, is maximal monotone and Lipschitzian with constant $1/\beta$ for every $\beta > 0$.

We shall use the following notation: $x_n \to x$ will mean that the sequence (x_n) converges strongly to x whereas $x_n \rightharpoonup x$ will mean that (x_n) converges weakly to x. The weak ω-limit set of a sequence (x_n) will be denoted by $\omega_w((x_n))$. That is,

$$\omega_w((x_n)) = \{x \in H : x_{n_k} \rightharpoonup x \text{ for some subsequence } (x_{n_k}) \text{ of } (x_n)\}.$$

We now recall some results which will be useful in proving our main results. We begin with an elementary property of norms in Hilbert spaces.

Lemma 1. *For all $x, y \in H$, we have*

$$\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y \rangle.$$

The next lemma is well known, it can be found in [16, p. 20].

Lemma 2. *Any maximal monotone operator $A : D(A) \subset H \to 2^H$ satisfies the demiclosedness principle. In other words, given any two sequences (x_n) and (y_n) satisfying $x_n \to x$ and $y_n \rightharpoonup y$ with $(x_n, y_n) \in G(A)$, then $(x, y) \in G(A)$.*

Lemma 3 (*Xu [17]*)**.** *For any $x \in H$ and $\mu \geq \beta > 0$,*

$$\left\| x - J_\beta^A x \right\| \leq 2 \left\| x - J_\mu^A x \right\|,$$

where $A : D(A) \subset H \to 2^H$ is a maximal monotone operator.

The proof of our main result is based on the following key lemmas.

Lemma 4 (*Maingé [18]*)**.** *Let (s_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence (s_{n_j}) of (s_n) such that $s_{n_j} < s_{n_j+1}$ for all $j \geq 0$. Define an integer sequence $(\tau(n))_{n\geq n_0}$ as*

$$\tau(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}. \tag{6}$$

Lemma 5 (*Boikanyo and Moroşanu [7]*)**.** *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)(1 - \lambda_n)s_n + \alpha_n b_n + \lambda_n c_n + d_n, \quad n \geq 0, \tag{7}$$

where (α_n), (λ_n), (b_n), (c_n) and (d_n) satisfy the conditions: (i) *$\alpha_n, \lambda_n \in [0, 1]$, with $\prod_{n=0}^{\infty}(1 - \alpha_n) = 0$,* (ii) *$\limsup_{n\to\infty} b_n \leq 0$,* (iii) *$\limsup_{n\to\infty} c_n \leq 0$, and* (iv) *$d_n \geq 0$ for all $n \geq 0$ with $\sum_{n=0}^{\infty} d_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.*

Remark 6. If $\lim_{n\to\infty} \alpha_n = 0$, then $\prod_{n=0}^{\infty}(1 - \alpha_n) = 0$ if and only if $\sum_{n=0}^{\infty} \alpha_n = \infty$.

3. The main results

We begin by proving a strong convergence result associated with the exact iterative process

$$v_{2n+1} = \alpha_n u + \delta_n v_{2n} + \gamma_n J_{\beta_n}^A v_{2n} \quad \text{for } n = 0, 1, \ldots, \tag{8}$$

$$v_{2n} = \lambda_n u + \rho_n v_{2n-1} + \sigma_n J_{\mu_n}^B v_{2n-1} \quad \text{for } n = 1, 2, \ldots, \tag{9}$$

where $\alpha_n, \delta_n, \gamma_n \in [0, 1]$ with $\alpha_n + \delta_n + \gamma_n = 1$, $\lambda_n, \rho_n, \sigma_n \in [0, 1]$ with $\lambda_n + \rho_n + \sigma_n = 1$ and $v_0, u \in H$ are given. The proof of the following theorem makes use of some ideas contained in [18,13,7,19].

Theorem 7. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $v_0, u \in H$, let (v_n) be the sequence generated by (8), (9), where $\alpha_n, \delta_n, \gamma_n \in [0, 1]$ with $\alpha_n + \delta_n + \gamma_n = 1$, $\lambda_n, \rho_n, \sigma_n \in [0, 1]$ with $\lambda_n + \rho_n + \sigma_n = 1$ and $\beta_n, \mu_n \in (0, \infty)$. Assume that* (i) *$\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \lambda_n = 0$,* (ii) *either $\sum_{n=0}^{\infty} \alpha_n = \infty$ or $\sum_{n=0}^{\infty} \lambda_n = \infty$,* (iii) *$\beta_n \geq \beta$ and $\mu_n \geq \mu$ for some $\beta, \mu > 0$, and* (iv) *$\gamma_n \geq \gamma$ and $\sigma_n \geq \sigma$ for some $\gamma, \sigma > 0$. Then (v_n) converges strongly to the point of F nearest to u.*

Proof. Let $p \in F$. Then from (9) and the fact that the resolvent operator of B is nonexpansive, we have

$$\begin{aligned} \|v_{2n} - p\| &\leq \lambda_n \|u - p\| + \rho_n \|v_{2n-1} - p\| + \sigma_n \left\| J_{\mu_n}^B v_{2n-1} - p \right\| \\ &\leq \lambda_n \|u - p\| + (1 - \lambda_n) \|v_{2n-1} - p\|. \end{aligned} \tag{10}$$

Similarly, from (8) and the fact that the resolvent operator of A is nonexpansive, we have

$$\begin{aligned} \|v_{2n+1} - p\| &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|v_{2n} - p\| \\ &\leq [\alpha_n + (1 - \alpha_n)\lambda_n] \|u - p\| + (1 - \alpha_n)(1 - \lambda_n) \|v_{2n-1} - p\| \\ &= [1 - (1 - \alpha_n)(1 - \lambda_n)] \|u - p\| + (1 - \alpha_n)(1 - \lambda_n) \|v_{2n-1} - p\|, \end{aligned}$$

where the last inequality follows from (10). Then by induction, it follows that

$$\|v_{2n+1} - p\| \leq \left[1 - \prod_{k=1}^{n} (1 - \alpha_k)(1 - \lambda_k) \right] \|u - p\| + \|v_1 - p\| \prod_{k=1}^{n} (1 - \alpha_k)(1 - \lambda_k).$$

This shows that the subsequence (v_{2n+1}) of (v_n) is bounded. In view of (10), the subsequence (v_{2n}) is also bounded. Hence the sequence (v_n) is bounded.

Note that the firm nonexpansiveness property of $J^A_{\beta_n}$ gives

$$\left\|J^A_{\beta_n} v_{2n} - p\right\|^2 \le \|v_{2n} - p\|^2 - \left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2.$$

Observe also that

$$\begin{aligned} 2\langle v_{2n} - p, J^A_{\beta_n} v_{2n} - p\rangle &= \|v_{2n} - p\|^2 + \left\|J^A_{\beta_n} v_{2n} - p\right\|^2 - \left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2 \\ &\le 2\left(\|v_{2n} - p\|^2 - \left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2\right). \end{aligned}$$

Again using the firm nonexpansiveness property of the resolvent operator, we see that

$$\begin{aligned} \left\|\delta_n(v_{2n} - p) + \gamma_n(J^A_{\beta_n} v_{2n} - p)\right\|^2 &= \delta_n^2 \|v_{2n} - p\|^2 + \gamma_n^2 \left\|J^A_{\beta_n} v_{2n} - p\right\|^2 + 2\gamma_n\delta_n\langle v_{2n} - p, J^A_{\beta_n} v_{2n} - p\rangle \\ &\le (1-\alpha_n)^2 \|v_{2n} - p\|^2 - \gamma_n(\gamma_n + 2\delta_n)\left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2. \end{aligned}$$

Moreover, from (8) and Lemma 1, we have

$$\begin{aligned} \|v_{2n+1} - p\|^2 &\le \left\|\delta_n(v_{2n} - p) + \gamma_n(J^A_{\beta_n} v_{2n} - p)\right\|^2 + 2\alpha_n\langle u - p, v_{2n+1} - p\rangle \\ &\le (1-\alpha_n)\|v_{2n} - p\|^2 - \varepsilon_1\left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2 + 2\alpha_n\langle u - p, v_{2n+1} - p\rangle, \end{aligned}$$

where $\varepsilon_1 > 0$ is such that $\gamma_n(\gamma_n + 2\delta_n) \ge \varepsilon_1$. Similarly, from (9) and Lemma 1, we get

$$\|v_{2n} - p\|^2 \le (1-\lambda_n)\|v_{2n-1} - p\|^2 - \varepsilon_2\left\|v_{2n-1} - J^B_{\mu_n} v_{2n-1}\right\|^2 + 2\lambda_n\langle u - p, v_{2n} - p\rangle, \tag{11}$$

where $\varepsilon_2 > 0$ is such that $\sigma_n(\sigma_n + 2\rho_n) \ge \varepsilon_2$. Therefore,

$$\begin{aligned} \|v_{2n+1} - p\|^2 &\le (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - p\|^2 - \varepsilon_1\left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2 + 2\alpha_n\langle u - p, v_{2n+1} - p\rangle \\ &\quad - \varepsilon_2(1-\alpha_n)\left\|v_{2n-1} - J^B_{\mu_n} v_{2n-1}\right\|^2 + 2\lambda_n(1-\alpha_n)\langle u - p, v_{2n} - p\rangle. \end{aligned} \tag{12}$$

Setting $s_n := \|v_{2n-1} - P_F u\|^2$, then we have for some positive constant M

$$s_{n+1} - s_n + \varepsilon_1\left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2 + \varepsilon_2\left\|v_{2n-1} - J^B_{\mu_n} v_{2n-1}\right\|^2 \le (\alpha_n + \lambda_n)M. \tag{13}$$

We shall show that $s_n \to 0$ as $n \to \infty$ by considering two possible cases for the sequence (s_n).

Case I: (s_n) is eventually decreasing (i.e., there exists $N \ge 0$ such that (s_n) is decreasing for all $n \ge N$). In this case, (s_n) is convergent. Then passing to the limit in (13), we get

$$\lim_{n\to\infty}\left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\| = 0 = \lim_{n\to\infty}\left\|v_{2n-1} - J^B_{\mu_n} v_{2n-1}\right\|. \tag{14}$$

Moreover, it follows from (8) that

$$\|v_{2n+1} - v_{2n}\| \le \alpha_n\|u - v_{2n}\| + \gamma_n\left\|J^A_{\beta_n} v_{2n} - v_{2n}\right\| \to 0,$$

as $n \to \infty$. Similarly, from (9), we have

$$\|v_{2n} - v_{2n-1}\| \le \lambda_n\|u - v_{2n-1}\| + \sigma_n\left\|J^B_{\mu_n} v_{2n-1} - v_{2n-1}\right\| \to 0,$$

as $n \to \infty$. Therefore, we derive

$$\lim_{n\to\infty}\|v_{n+1} - v_n\| = 0. \tag{15}$$

On the other hand, from Lemma 3 and the first part of (14), we have

$$\left\|v_{2n} - J^A_{\beta} v_{2n}\right\| \le 2\left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\| \to 0 \quad \text{as } n \to \infty.$$

Since A_β^{-1}, where A_β denotes the Yosida approximation of A, is demiclosed, it follows that $\omega_w((v_{2n})) \subset A^{-1}(0)$. Similarly, from the second part of (14) and Lemma 3,

$$\left\|v_{2n-1} - J^B_{\mu} v_{2n-1}\right\| \le 2\left\|v_{2n-1} - J^B_{\mu_n} v_{2n-1}\right\| \to 0 \quad \text{as } n \to \infty.$$

From the demiclosedness property of the Yosida approximation of the operator B, one derives $\omega_w((v_{2n+1})) \subset B^{-1}(0)$. Moreover, in view of (15) we get $\omega_w((v_n)) \subset F := A^{-1}(0) \cap B^{-1}(0)$. Therefore, there exists a subsequence (v_{n_k}) of (v_n) converging weakly to some $z \in F$ such that

$$\begin{aligned} \limsup_{n\to\infty}\langle u - P_F u, v_n - P_F u\rangle &= \lim_{k\to\infty}\langle u - P_F u, v_{n_k} - P_F u\rangle \\ &= \langle u - P_F u, z - P_F u\rangle \le 0, \end{aligned}$$

where the inequality above follows from one of the basic properties of projections. Now, replacing p by $P_F u$ in (12) gives

$$\|v_{2n+1} - P_F u\|^2 \leq (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - P_F u\|^2 + 2\alpha_n\langle u - P_F u, v_{2n+1} - P_F u\rangle + 2\lambda_n(1-\alpha_n)\langle u - P_F u, v_{2n} - P_F u\rangle. \quad (16)$$

Using Lemma 5 we get $\|v_{2n+1} - P_F u\| \to 0$ as $n \to \infty$. Passing to the limit in (11), we also get $\|v_{2n} - P_F u\| \to 0$ as $n \to \infty$. Therefore, $\|v_n - P_F u\| \to 0$ as $n \to \infty$.

Case II: (s_n) is not eventually decreasing, that is, there is a subsequence (s_{n_j}) of (s_n) such that $s_{n_j} < s_{n_j+1}$ for all $j \geq 0$. Define an integer sequence $(\tau(n))_{n\geq n_0}$ as in Lemma 4. Since $s_{\tau(n)} \leq s_{\tau(n)+1}$ for all $n \geq n_0$, it follows from (13) that

$$\lim_{n\to\infty}\left\|v_{2\tau(n)} - J^A_{\beta_{\tau(n)}} v_{2\tau(n)}\right\| = 0 = \lim_{n\to\infty}\left\|v_{2\tau(n)-1} - J^B_{\mu_{\tau(n)}} v_{2\tau(n)-1}\right\|. \quad (17)$$

As in Case I, we derive $\omega_w((v_{2\tau(n)-1})) \subset B^{-1}(0)$ and $\omega_w((v_{2\tau(n)})) \subset A^{-1}(0)$. Since we have from (9) and the second part of (17)

$$\left\|v_{2\tau(n)} - v_{2\tau(n)-1}\right\| \leq \lambda_{\tau(n)}\left\|u - v_{2\tau(n)-1}\right\| + \sigma_{\tau(n)}\left\|J^B_{\mu_{\tau(n)}} v_{2\tau(n)-1} - v_{2\tau(n)-1}\right\| \to 0,$$

as $n \to \infty$, it follows that $\omega_w((v_{2\tau(n)})) \subset F$. Consequently,

$$\limsup_{n\to\infty}\langle u - P_F u, v_{2\tau(n)} - P_F u\rangle \leq 0. \quad (18)$$

Note that from (16), we have

$$\|v_{2n+1} - P_F u\|^2 \leq (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - P_F u\|^2 + \alpha_n K\|v_{2n+1} - v_{2n}\| + 2[\lambda_n(1-\alpha_n) + \alpha_n]\langle u - P_F u, v_{2n} - P_F u\rangle,$$

where $K := \|u - P_F u\|$. Therefore,

$$s_{\tau(n)+1} \leq (1-\alpha_{\tau(n)})(1-\lambda_{\tau(n)})s_{\tau(n)} + \alpha_{\tau(n)}K\left\|v_{2\tau(n)+1} - v_{2\tau(n)}\right\| + 2(\lambda_{\tau(n)}(1-\alpha_{\tau(n)}) + \alpha_{\tau(n)})\langle u - P_F u, v_{2\tau(n)} - P_F u\rangle.$$

Since $s_{\tau(n)} \leq s_{\tau(n)+1}$ for all $n \geq n_0$ (cf. Lemma 4), we have

$$s_{\tau(n)+1} \leq 2\langle u - P_F u, v_{2\tau(n)} - P_F u\rangle + \frac{\alpha_{\tau(n)}K\left\|v_{2\tau(n)+1} - v_{2\tau(n)}\right\|}{\lambda_{\tau(n)}(1-\alpha_{\tau(n)}) + \alpha_{\tau(n)}} \leq 2\langle u - P_F u, v_{2\tau(n)} - P_F u\rangle + K\left\|v_{2\tau(n)+1} - v_{2\tau(n)}\right\|, \quad (19)$$

for all $n \geq n_0$. Now from (8), we have

$$\left\|v_{2\tau(n)+1} - v_{2\tau(n)}\right\| \leq \alpha_{\tau(n)}\left\|u - v_{2\tau(n)}\right\| + \gamma_{\tau(n)}\left\|J^A_{\beta_{\tau(n)}} v_{2\tau(n)} - v_{2\tau(n)}\right\| \to 0, \quad (20)$$

as $n \to \infty$. Using (18) and (20) in (19), we get $s_{\tau(n)+1} \to 0$ as $n \to \infty$. Hence from (6) it follows that $s_n \to 0$ as $n \to \infty$. That is, $v_{2n+1} \to P_F u$ as $n \to \infty$. Note that from (11), we have

$$\|v_{2n} - P_F u\|^2 \leq (1-\lambda_n)\|v_{2n-1} - P_F u\|^2 + \lambda_n M,$$

which implies that $v_{2n} \to P_F u$ as $n \to \infty$. Therefore, $v_n \to P_F u$ as $n \to \infty$. This completes the proof of the theorem. □

We now state and prove a strong convergence result for sequences generated from algorithm (4), (5) under several alternative conditions on the error sequences (e_n) and (e'_n).

Theorem 8. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $x_0, u \in H$, let (x_n) be the sequence generated by (4), (5), where $\alpha_n, \delta_n, \gamma_n \in (0,1)$ with $\alpha_n + \delta_n + \gamma_n = 1$, $\lambda_n, \rho_n, \sigma_n \in (0,1)$ with $\lambda_n + \rho_n + \sigma_n = 1$ and $\beta_n, \mu_n \in (0,\infty)$. Assume that* (i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\lim_{n\to\infty}\lambda_n = 0$, (ii) *either* $\sum_{n=0}^{\infty}\alpha_n = \infty$ *or* $\sum_{n=0}^{\infty}\lambda_n = \infty$, (iii) $\beta_n \geq \beta$ *and* $\mu_n \geq \mu$ *for some* $\beta, \mu > 0$, *and* (iv) $\gamma_n \geq \gamma$ *and* $\sigma_n \geq \sigma$ *for some* $\gamma, \sigma > 0$. *In addition, if any of the following conditions is satisfied:*

(a) $\sum_{n=0}^{\infty}\|e_n\| < \infty$ *and* $\sum_{n=1}^{\infty}\|e'_n\| < \infty$;
(b) $\sum_{n=0}^{\infty}\|e_n\| < \infty$ *and* $\|e'_n\|/\alpha_n \to 0$;
(c) $\sum_{n=0}^{\infty}\|e_n\| < \infty$ *and* $\|e'_n\|/\lambda_n \to 0$;
(d) $\|e_n\|/\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty}\|e'_n\| < \infty$;
(e) $\|e_n\|/\lambda_n \to 0$ *and* $\sum_{n=1}^{\infty}\|e'_n\| < \infty$;
(f) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\alpha_n \to 0$;
(g) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$;

(h) $\|e_n\|/\lambda_n \to 0$ *and* $\|e'_n\|/\alpha_n \to 0$*;*
(i) $\|e_n\|/\lambda_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$*;*
(j) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$*;*
(k) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$*;*
(l) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$*;*
(m) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$*;*
(n) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$*,*

then (x_n) *converges strongly to the point of* F *nearest to* u*.*

Proof. Taking Theorem 7 into account, it is enough to prove that $\|x_n - v_n\| \to 0$ as $n \to \infty$. Since the resolvent of A is nonexpansive, we derive from (4) and (8) that

$$\begin{aligned} \|x_{2n+1} - v_{2n+1}\| &\le \delta_n \|x_{2n} - v_{2n}\| + \gamma_n \left\|J^A_{\beta_n} x_{2n} - J^A_{\beta_n} v_{2n}\right\| + \|e_n\| \\ &\le (1-\alpha_n)\|x_{2n} - v_{2n}\| + \|e_n\| . \end{aligned} \tag{21}$$

Similarly, from (5) and (9), we have

$$\|x_{2n} - v_{2n}\| \le (1-\lambda_n)\|x_{2n-1} - v_{2n-1}\| + \left\|e'_n\right\| . \tag{22}$$

These two inequalities imply that

$$\|x_{2n+1} - v_{2n+1}\| \le (1-\alpha_n)(1-\lambda_n)\|x_{2n-1} - v_{2n-1}\| + \|e_n\| + \left\|e'_n\right\| .$$

Therefore if the sequence of errors satisfy any of the conditions (a)–(i), then it readily follows from Lemma 5 that $\|x_{2n+1} - v_{2n+1}\| \to 0$ as $n \to \infty$. Passing to the limit in (22), we derive $\|x_{2n} - v_{2n}\| \to 0$ as well. If the sequence of errors satisfy any of the conditions (j)–(n), then we get from (21) and (22)

$$\|x_{2n} - v_{2n}\| \le (1-\alpha_{n-1})(1-\lambda_n)\|x_{2n-2} - v_{2n-2}\| + \|e_{n-1}\| + \left\|e'_n\right\| .$$

It then follows from Lemma 5 that $\|x_{2n} - v_{2n}\| \to 0$ as $n \to \infty$. Passing to the limit in (21), we derive $\|x_{2n+1} - v_{2n+1}\| \to 0$ as well. This completes the proof of the theorem. □

Remark 9. Algorithm (4), (5) contains several algorithms as special cases. For instance, setting $\delta_n = 0 = \rho_n$, we get

$$x_{2n+1} = \alpha_n u + (1-\alpha_n) J^A_{\beta_n} x_{2n} + e_n \quad \text{for } n = 0, 1, \dots, \tag{23}$$

$$x_{2n} = \lambda_n u + (1-\lambda_n) J^B_{\mu_n} x_{2n-1} + e'_n \quad \text{for } n = 1, 2, \dots, \tag{24}$$

which converges strongly if the conditions listed in the following corollary are met.

Corollary 10. *Let* $A : D(A) \subset H \to 2^H$ *and* $B : D(B) \subset H \to 2^H$ *be maximal monotone operators with* $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$*. For arbitrary but fixed vectors* $x_0, u \in H$*, let* (x_n) *be the sequence generated by* (23)*,* (24)*, where* $\alpha_n, \lambda_n \in (0,1)$ *and* $\beta_n, \mu_n \in (0,\infty)$*. Assume that* (i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \lambda_n = 0$*,* (ii) *either* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *or* $\sum_{n=0}^{\infty} \lambda_n = \infty$*, and* (iii) *both* (β_n) *and* (μ_n) *are bounded below away from zero. In addition, if any of the conditions* (a)–(n) *in* Theorem 8 *is satisfied, then* (x_n) *converges strongly to the point of* F *nearest to* u*.*

Remark 11. Note that for (4) and (5) with $A = \partial I_{K_1}$, the subdifferential of the indicator function I_{K_1} on K_1, and $B = \partial I_{K_2}$, where K_1 and K_2 are nonempty, closed and convex subsets of H, then Algorithm (4), (5) reduces to

$$\begin{aligned} &x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n P_{K_1} x_{2n} + e_n \quad \text{for } n = 0, 1, \dots, \\ &x_{2n} = \lambda_n u + \rho_n x_{2n-1} + \sigma_n P_{K_2} x_{2n-1} + e'_n \quad \text{for } n = 1, 2, \dots, \end{aligned}$$

which converges strongly according to Theorem 8 to a point in $K_1 \cap K_2$ nearest to u. One can say that the above algorithm is the modified method of alternating projections which always produce sequences that converge strongly, unlike the original method of alternating projection introduced by von Neumann which fails to converge strongly in general [4,5].

Remark 12. Another important algorithm which can be derived from Algorithm (4), (5) is the contraction proximal point algorithm (CPPA). Indeed, setting $\lambda_n = 0$ and $B = \partial I_K$, where $K = H$, then for $x_n := x_{2n+1}$, we reobtain the following generalized CPPA:

$$x_{n+1} = \alpha_n u + \delta_n x_n + \gamma_n J^A_{\beta_n} x_n + e_n \quad \text{for } n = 0, 1, \dots,$$

which was introduced by Yao and Noor [12]. Therefore, Theorem 8 contains as special cases [12, Theorem 3.3], [13, Theorem 1], [14, Theorems 2–6], [20, Theorems 1–4] and many other results announced recently.

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