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Elliptic-like regularization of semilinear evolution equations

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ABSTRACT

Consider in a real Hilbert space the Cauchy problem (P_0): $u'(t)+Au(t)+Bu(t)=f(t),\ 0\le t\le T;\ u(0)=u_0$, where $-A$ is the generator of a C_0-semigroup of linear contractions and B is a smooth nonlinear operator. We associate with (P_0) the following problem: (P_1^ε): $-\varepsilon u''(t)+u'(t)+Au(t)+Bu(t)=f(t),\ 0\le t\le T;\ u(0)=u_0,\ u(T)=u_1$, where $\varepsilon>0$ is a small parameter. Existence, uniqueness and higher regularity for both (P_0) and (P_1^ε) are investigated and an asymptotic expansion for the solution of problem (P_1^ε) is established, showing the presence of a boundary layer near $t=T$.

1. Introduction

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$. Denote by (P_0) the following Cauchy problem

$$\begin{cases} u'(t)+Au(t)+Bu(t)=f(t), & 0\le t\le T, \quad (E)\\ u(0)=u_0, & \quad (IC)\end{cases} \tag{P_0}$$

where $T>0$ is a given time instant, $u_0\in H$ is a given initial state, $f\colon[0,T]\to H$, $A\colon D(A)\subset H\to H$ is a linear operator, such that

(Hyp1) $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, say $\{S(t)\colon H\to H;\ t\ge 0\}$.
(Hyp2) $B\colon H\to H$ is a nonlinear, Fréchet differentiable operator satisfying $\sup_{x\in H}\|B'(x)\|_{L(H)}=C<\infty$, where $B'(x)$ is the Fréchet derivative of B at $x\in H$, and $L(H)$ denotes the space of linear continuous operators from H into H, equipped with the usual operator norm $\|\cdot\|_{L(H)}$.

So equation (E) is a semilinear one.

In what follows we recall some basic definitions that are needed in this paper.

A family $\{S(t);\ t\ge 0\}\subset L(H)$ is called a *semigroup* (of linear continuous operators on H) if

(i) $S(0)=I$ (the identity operator on H).
(ii) $S(t+s)=S(t)S(s)$ for all $t,s\ge 0$.

If, in addition,

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(iii) $\lim_{t\to 0^+} \|S(t)x - x\| = 0$ for all $x \in H$,

then $\{S(t);\ t \geq 0\}$ is said to be a C_0-*semigroup*. If, moreover, $S(t)$ is a contraction (i.e., $\|S(t)\|_{L(H)} \leq 1$) for all $t \geq 0$ then $\{S(t);\ t \geq 0\}$ is called a C_0-*semigroup of contractions*.

Given a C_0-semigroup $\{S(t);\ t \geq 0\} \subset L(H)$ its *infinitesimal generator* $A: D(A) \subset H \to H$ is defined by

$$D(A) = \left\{x \in H\colon \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists}\right\},$$

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t}.$$

For more information on C_0-semigroups we refer the reader to [1,2].

A set $S \subset H \times H$ is said to be *monotone* if

$$(x_1 - x_2, y_1 - y_2) \geq 0 \quad \forall (x_1, y_1), (x_2, y_2) \in S.$$

A nonlinear operator $Q: D(Q) \subset H \to H$ is called *monotone* if its graph is a monotone subset of $H \times H$, i.e.,

$$(x_1 - x_2, Qx_1 - Qx_2) \geq 0, \quad \forall x_1, x_2 \in D(Q).$$

A monotone nonlinear operator $Q: D(Q) \subset H \to H$ is said to be *maximal monotone* if its graph is not properly included in any monotone subset of $H \times H$.

For information on the theory of monotone operators (including the multivalued case) we refer the reader to the monographs [3,4]. In particular, we recall that if $Q: D(Q) \subset H \to H$ is a maximal monotone operator, then the range of $I + \lambda Q$ is the whole H, $I + \lambda Q$ is invertible and its inverse is Lipschitzian with the Lipschitz constant equal to 1, for all $\lambda > 0$. It is well known that if $A: D(A) \subset H \to H$ satisfies (Hyp 1), then A is maximal monotone.

Let $Q: D(Q) \subset H \to H$ be a maximal monotone operator. For every $\lambda > 0$, set

$$J_\lambda = (I + \lambda Q)^{-1} \quad \text{and} \quad Q_\lambda = \frac{1}{\lambda}(I - J_\lambda),$$

where J_λ is called the *resolvent* of Q, and Q_λ is the *Yosida approximation* of Q.

Now consider the second order equation

$$-\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \quad 0 \leq t \leq T, \tag{E_ε}$$

where $\varepsilon > 0$ is a small parameter. In this paper we intend to investigate whether the solution of (P_0) could be approximated by a solution of (E_ε) which is expected to be more regular. Thus, it is natural to associate with (E_ε) the same condition (IC). Since the problem (E_ε), (IC) is incomplete, we need to associate with (E_ε) an additional condition. Let it be

$$u(T) = u_1. \tag{C}$$

So we have the problem

$$(E_\varepsilon),\ (IC),\ (C). \tag{P_1^ε}$$

A typical example.

Let $H = L^2(\Omega)$ with the usual scalar product and norm, where Ω is a bounded open subset of $\mathbb{R}^3$ with a sufficiently smooth boundary $\partial\Omega$,

$$A = -\Delta \quad \text{with } D(A) = H_0^1(\Omega) \cap H^2(\Omega),$$

$$Bu = b \circ u,$$

where $b: \mathbb{R} \to \mathbb{R}$ is a differentiable function, with $0 \leq b'(r) \leq C < \infty$ $\forall r \in \mathbb{R}$. In this case, (E) is the nonlinear heat equation

$$u_t - \Delta_x u + b \circ u = f(x, t), \quad x \in \Omega,\ 0 \leq t \leq T,$$

while (E_ε) is an elliptic type equation. Thus it is natural to call (E_ε) an elliptic-like regularization of (E).

It is worth pointing out A. Perjan's work on the behavior of solutions of the hyperbolic-like problem $\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t)$, $0 \leq t \leq T$, $u(0) = u_0$, $u'(0) = u_1$ in a Hilbert space H as $\varepsilon \to 0$, where A is a linear, symmetric, strongly positive operator and B is a nonlinear operator. See, e.g., [5,6]. See also [7], Chapter 11, for a particular case, when $H = L^2(\Omega)$.

The rest of this paper is organized as follows: Sections 2 and 3 are concerned with the existence, uniqueness, and regularity of the solutions of problems (P_0) and (P_1^ε). While the results of these sections are important in themselves, we need them in Section 4 in order to validate the asymptotic expansion for the solution of problem (P_1^ε) that will be first derived heuristically.

It is worth mentioning that our analysis covers many applications, in particular the initial-boundary value problem for the semilinear heat equation that is discussed in our paper. Recall that the linear case of (P_1^ε) (i.e., the case $B = 0$) was discussed by Lions [8], p. 407, who called it an *elliptic-evolution problem*. Some examples were also provided there. Lions explained (see [8], p. IX) that sometimes it might be useful to consider regularizations of problem (P_0), including (P_1^ε), that provide good solutions approximating the solution of (P_0) for ε small. This regularization method was also called by Lions the method of artificial viscosity (due to the additional term involving ε).

2. Existence and regularity for problem (P_0)

Throughout this paper we adopt the usual definitions of a strong and weak solution for the Cauchy problem associated with a nonlinear operator $Q: D(Q) \subset H \to H$,

$$u'(t) + Qu(t) = f(t), \quad 0 \le t \le T, \qquad u(0) = u_0. \tag{P}$$

Definition 2.1. Let $f \in L^1(0, T; H)$. A function $u \in C([0, T]; H)$ is said to be a strong solution of problem (P) if: u is absolutely continuous on every compact subinterval of $(0, T)$; $u(0) = u_0$ and u satisfies the above equation of (P) for a.e. $t \in (0, T)$.

Definition 2.2. A function $u \in C([0, T]; H)$ is said to be a weak solution of problem (P) if there exist sequences $\{u_n\} \subset C([0, T]; H)$ and $\{f_n\} \subset L^1(0, T; H)$ such that each u_n is absolutely continuous on every compact subinterval of $(0, T)$ and $u_n'(t) + Qu_n(t) = f_n(t)$ a.e. $t \in (0, T)$, for each n; $u_n \to u$ in $C([0, T]; H)$; $u(0) = u_0$; and $f_n \to f$ in $L^1(0, T; H)$.

Lemma 2.3. *Let X be a Banach space, and $B: X \to X$ be differentiable satisfying $\sup_{x\in X} \|B'(x)\|_{L(X)} = C < \infty$, then*

$$\|Bx - By\| \le C\|x - y\| \quad \forall x, y \in X. \tag{2.1}$$

Proof. The proof is not new but we give it for the convenience of the reader. Let $f \in X^*$ be arbitrary. Consider $g: [0, 1] \to \mathbb{R}$ defined as

$$g(t) = (f \circ B)(y + t(x - y)).$$

By the mean value theorem for real valued functions, there exists a $c \in (0, 1)$ such that

$$g(1) - g(0) = g'(c).$$

So,

$$f(Bx) - f(By) = f[B'(y + c(x - y))(x - y)]$$
$$\Rightarrow f(Bx - By) = f[B'(y + c(x - y))(x - y)].$$

If $Bx - By = 0$, then (2.1) is trivial. Assume that $Bx - By \ne 0$, then by the Hahn–Banach theorem, there exists $f \in X^*$ satisfying $\|f\| = 1$, and $f(Bx - By) = \|Bx - By\|$. From this (2.1) follows easily. □

Remark 2.4. If B satisfies (Hyp 2), then by Lemma 2.3 B is a Lipschitz operator. If in addition B is assumed to be monotone, then B is maximal monotone and since A is maximal monotone (cf. (Hyp 1)), it follows that $A + B$ is maximal monotone as well.

Now we are going to state and prove some existence results for problem (P_0). For the convenience of the reader, we first recall some known existence results.

Lemma 2.5 (*See, e.g., [4], Theorem 2.1, p. 48 and Remark 2.1, p. 53*). *If $Q: D(Q) \subset H \to H$ is maximal monotone, $u_0 \in D(Q)$ and $f \in W^{1,1}(0, T; H)$, then problem (P) has a unique strong solution $u \in W^{1,\infty}(0, T; H)$. The conclusion still holds if Q is replaced by $Q + Q_1$, where $Q_1: D(Q_1) = H \to H$ is a Lipschitz operator.*

Lemma 2.6 (*Brezis; See, e.g., [4], Theorem 2.4, p. 56*). *If Q is the subdifferential of a proper, convex, lower semicontinuous function $\varphi: H \to (-\infty, +\infty]$, $u_0 \in \overline{D(Q)}$ and $f \in L^2(0, T; H)$, then problem (P) has a unique strong solution u, such that $t^{1/2}u' \in L^2(0, T; H)$. If, in addition, $\varphi(u_0) < +\infty$, then $u' \in L^2(0, T; H)$.*

Lemma 2.7 (*See, e.g., [7], Theorem 2.0.28, p. 30*). *If $Q := A + F(t, \cdot)$, where A is a linear operator satisfying* (Hyp1) , *$F(\cdot, z) \in L^1(0, T; H)$ for all $z \in H$, and there exists a constant $\omega > 0$ such that*

$$\|F(t, z_1) - F(t, z_2)\| \le \omega\|z_1 - z_2\| \quad \forall t \in [0, T],\ z_1, z_2 \in H,$$

then, for every $u_0 \in H$, problem (P) has a unique mild solution $u \in C([0, T]; H)$, i.e.,

$$u(t) = S(t)u_0 - \int_0^t S(t - s)F(s, u(s))\, ds, \quad \forall t \in [0, T].$$

Lemma 2.8 (*Brezis [3], pp. 106–107*). *Assume that all the assumptions of Lemma 2.7 hold. If, in addition, A is self-adjoint and $F(\cdot, z) \in L^2(0, T; H)$ for all $z \in H$, then u is a strong solution of problem (P), such that $t^{1/2}u' \in L^2(0, T; H)$.*

Lemma 2.9 (*Kato [9]*). *Assume that $Q = Q(t,\cdot)$ (i.e., Q is time dependent), $Q(t,\cdot)$ is single-valued, maximal monotone, with $D(Q(t,\cdot)) = D$ for all $t \in [0,T]$ (i.e., $D(Q(t,\cdot))$ is independent of t), and the following condition is satisfied*

$$\|Q(t,z) - Q(s,z)\| \le L|t-s|(1 + \|z\| + \|Q(s,z)\|),$$

for all $z \in D$, $s,t \in [0,T]$, where L is a positive constant. Then, for every $u_0 \in D$, problem (P) has a unique strong solution $u \in W^{1,\infty}(0,T;\ H)$.

Theorem 2.10. *Assume* (Hyp1) *and* (Hyp2). *If $u_0 \in H$ and $f \in L^1(0,T;H)$, then (P_0) has a unique mild solution $u \in C([0,T];H)$, i.e.,*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)[f(s) - Bu(s)]\,ds, \quad 0 \le t \le T. \tag{2.2}$$

If $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$ then $u \in C^1([0,T];H)$ and it is a strong solution of (P_0), satisfying (E) of (P_0) for all $t \in [0,T]$.

Proof. The first part follows by Lemma 2.7. Now, if $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$, then according to Lemma 2.5, the mild solution u belongs to $W^{1,\infty}(0,T;H)$ and it is a strong solution. Moreover, since

$$u'(t) = S(t)(f(0) - Au_0 - Bu_0) + \int_0^t S(t-s)[f'(s) - (Bu)'(s)]\,ds, \quad 0 \le t \le T, \tag{2.3}$$

it follows that $u' \in C([0,T];H)$. This completes the proof. □

Remark 2.11. If $u_0 \in H$ and $f \in L^1(0,T;H)$ then the mild solution u of problem (P_0) is also a weak solution. Indeed, let $(u_0^n, f_n) \in D(A) \times W^{1,1}(0,T;H)$ approximate (u_0, f) in $H \times L^1(0,T;H)$. Denote by u_n the strong solution of (P_0) with $u_0 := u_0^n$ and $f := f_n$. Then,

$$u_n(t) = S(t)u_0^n + \int_0^t S(t-s)[f_n(s) - Bu_n(s)]\,ds, \quad 0 \le t \le T. \tag{2.4}$$

Therefore, since $S(t)$ is a contraction for each $t \ge 0$ and B is Lipschitzian, we have

$$\|u_n(t) - u_m(t)\| \le \|u_0^n - u_0^m\| + \|f_n - f_m\|_{L^1(0,T;H)} + C\int_0^t \|u_n(s) - u_m(s)\|\,ds,$$

for all $t \in [0,T]$.

It follows by Gronwall's Lemma that

$$\|u_n(t) - u_m(t)\| \le (\|u_0^n - u_0^m\| + \|f_n - f_m\|_{L^1(0,T;H)})e^{Ct}, \quad \forall t \in [0,T].$$

This shows that u_n converges in $C([0,T];H)$ and its limit $\tilde{u}$ is a weak solution of problem (P_0). By passing to the limit in (2.4) we can see that $\tilde{u}$ is also a mild solution of the same problem, so by the uniqueness of the mild solution $\tilde{u} = u$.

Theorem 2.12. *Assume that* (Hyp1) *and* (Hyp2) *hold and, in addition, that A is self-adjoint. If $u_0 \in H$ and $f \in L^2(0,T;H)$, then problem (P_0) has a unique strong solution u, such that $t^{1/2}u' \in L^2(0,T;H)$.*

Proof. The result follows easily by Lemma 2.8. □

Theorem 2.13 (*Higher Regularity*). *Assume that* (Hyp1) *and* (Hyp2) *hold and, in addition, that A is self-adjoint. If $u_0 \in D(A)$ and $f \in W^{1,2}(0,T;H)$, then the solution u of problem (P_0) belongs to $C^1([0,T];H)$ and u' is differentiable, i.e., with $t^{1/2}u'' \in L^2(0,T;H)$. If in addition $f(0) - Au_0 - Bu_0 \in D(A^{1/2})$, then $u \in W^{2,2}(0,T;H)$.*

Proof. If $u_0 \in D(A)$ and $f \in W^{1,2}(0,T;H)$ it follows by Theorem 2.10 that $u \in C^1([0,T];H)$. Obviously u satisfies the equation

$$u'(t) = S(t)(f(0) - Au_0 - Bu_0) + \int_0^t S(t-s)[f'(s) - B'(u(s))u'(s)]\,ds \tag{2.5}$$

for all $t \in [0,T]$. Now, consider the equation (obtained from (E) by formal differentiation)

$$\begin{cases} v'(t) + Av(t) + B'(u(t))v(t) = f'(t), & 0 \le t \le T, \\ v(0) = f(0) - Au_0 - Bu_0. \end{cases} \tag{CP}$$

(*CP*) has a mild solution $v = v(t) \in C([0, T]; H)$,

$$v(t) = S(t)(f(0) - Au_0 - Bu_0) + \int_0^t S(t-s)[f'(s) - B'(u(s))v(s)]\, ds, \tag{2.6}$$

for all $t \in [0, T]$. From (2.5) and (2.6) we derive

$$\|v(t) - u'(t)\| \le \int_0^t \|B'(u(s))\|_{L(H)} \cdot \|v(s) - u'(s)\| ds, \quad 0 \le t \le T$$

which implies $v(t) = u'(t)$ for all $0 \le t \le T$.

In fact, since A is self-adjoint, the above Cauchy problem (*CP*) has a strong solution v, with $\sqrt{t}\, v' \in L^2(0, T; H)$ (cf. Lemma 2.8). Therefore, $\sqrt{t}\, u'' \in L^2(0, T; H)$. Now, if in addition $f(0) - Au_0 - Bu_0 \in D(A^{1/2})$, then the solution $v = u'$ of problem (*CP*) belongs to $W^{1,2}(0, T; H)$. This follows by Lemma 2.6, where $\varphi(x) = (1/2)\|A^{1/2}x\|^2$ for $x \in D(A^{1/2})$, and $\varphi(x) = +\infty$ for $x \in H \setminus D(A^{1/2})$. So the proof is complete. □

If A is not self-adjoint, then a higher regularity result holds under more restrictive conditions, as shown in the next theorem.

Theorem 2.14. *Assume* (Hyp1) *and* (Hyp2). *If* $f \in W^{2,\infty}(0, T; H)$, $u_0 \in D(A)$, $f(0) - Au_0 - Bu_0 \in D(A)$, *and B is twice differentiable with B'' bounded on bounded sets, then problem* (P_0) *has a unique solution* $u \in C^2([0, T]; H)$.

Proof. Taking into account Theorem 2.10, it suffices to prove that the above (*CP*) has a solution $v \in C^1([0, T]; H)$. Since $v = u'$ this would conclude the proof. We will apply Kato's theorem (Lemma 2.9). To this purpose, let us replace (*CP*) by an equivalent one which fits in the framework of Kato's theorem. Let $C_1 > C$. Multiply the equation (*CP*) by $e^{-C_1 t}$. Denoting $w(t) = e^{-C_1 t} v(t)$, we obtain the following Cauchy problem (which is equivalent to (*CP*))

$$\begin{cases} w'(t) + Aw(t) + (C_1 I + B'(u(t)))w(t) = e^{-C_1 t} f'(t), & 0 \le t \le T, \\ w(0) = f(0) - Au_0 - Bu_0. \end{cases} \tag{$\widetilde{CP}$}$$

The operator $A + C_1 I + B'(u(t))$ is maximal monotone for all $t \in [0, T]$ and its domain is $D(A)$ for all $t \in [0, T]$. Thus, by Kato's existence result, ($\widetilde{CP}$) has a unique solution $w \in W^{1,\infty}(0, T; H)$. Therefore, $v = u' \in W^{1,\infty}(0, T; H)$. In fact, $v = u'$ is a mild solution of problem (*CP*) (see (2.6)) and satisfies

$$u''(t) = v'(t) = S(t)v'(0) + \int_0^t S(t-s)[f''(s) - B''(u(s))u'(s)v(s) - B'(u(s))v'(s)]ds, \quad 0 \le t \le T$$

which shows that $u'' \in C([0, T]; H)$. This concludes the proof. □

Remark 2.15. It is worth pointing out that by using the above ideas every level of regularity for the solution of (P_0) can be reached under appropriate conditions.

3. Existence and regularity for problem (P_1^ε)

In this section we assume $\varepsilon = 1$ without any loss of generality. So (P_1^ε) becomes

$$\begin{cases} -u'' + u' + Au + Bu = f(t), & 0 \le t \le T, \\ u(0) = u_0, \quad u(T) = u_1. \end{cases} \tag{P_1}$$

Theorem 3.1. *If* (Hyp1), (Hyp2) *hold, B is monotone,* $f \in L^2(0, T; H)$, *and* $u_0, u_1 \in D(A)$, *then problem* (P_1) *has a unique solution* $u \in W^{2,2}(0, T; H)$.

Proof. $Q = A + B$ is a maximal monotone operator, with $D(Q) = D(A)$ (see Remark 2.4 above). Therefore, one can apply [10] to derive the existence of at least one solution $u \in W^{2,2}(0, T; H)$ for problem (P_1). Indeed, all conditions in [10] are fulfilled, with $\alpha = \beta =$ the subdifferential of the indicator function φ of the set $\{0\} \subset H$, i.e., $\varphi(0) = 0$ and $\varphi(x) = +\infty$ for all $x \in H \setminus \{0\}$.

In fact, u is unique. Indeed, if v is another solution of (P_1), then

$$\begin{cases} -(u-v)'' + (u-v)' + A(u-v) + Bu - Bv = 0, & 0 \le t \le T, \\ (u-v)(0) = 0, \qquad (u-v)(T) = 0. \end{cases}$$

If we multiply by $(u - v)$ the above equation and use the monotonicity of Q, we obtain

$$-\int_0^T ((u-v)'', u-v)dt + \int_0^T ((u-v)', u-v)dt \le 0.$$

This implies

$$\int_0^T \|u'-v'\|^2 dt + \underbrace{\frac{1}{2}\|u-v\|^2 \Big|_{t=0}^{t=T}}_{=0} \le 0,$$

hence $u'-v' \equiv 0$, i.e., $u-v$ is a constant function. In fact $u \equiv v$ since $u(0)=v(0)$. □

Theorem 3.2. *Assume* (Hyp1) *and* (Hyp2) *hold and B is monotone. If* $f \in L^2(0,T;H)$ *and* $u_0, u_1 \in H$*, then problem* (P_1) *has a unique solution* $u \in C([0,T];H) \cap W^{2,2}_{loc}(0,T;H)$*, with* $t^{1/2}(T-t)^{1/2}u', t^{3/2}(T-t)^{3/2}u'' \in L^2(0,T;H)$.

Proof. Note that $\overline{D(A)} = \overline{D(Q)} = H$, where $Q = A+B$. We will use a technique similar to that of [11]. We continue the proof with the following claim. □

Claim 1. *If* u, v *are two solutions of* (P_1) *with the properties specified in the statement of Theorem 3.2, then* $t \mapsto e^{-t}\|u(t)-v(t)\|^2$ *is a convex function on* $[0,T]$*, and*

$$\|u-v\|_{C([0,T];H)} \le \max\{e^{T/2}\|u(0)-v(0)\|, \|u(T)-v(T)\|\}, \tag{3.1}$$

$$\int_0^T t(T-t)\|u'-v'\|^2 dt \le 2T(e^T\|u(0)-v(0)\|^2 + \|u(T)-v(T)\|^2). \tag{3.2}$$

Proof of Claim 1. Let

$$g(t) = \frac{1}{2}\|u(t)-v(t)\|^2, \quad 0 \le t \le T.$$

Obviously, $g \in C([0,T])$ and

$$g'' = (u''-v'', u-v) + \|u'-v'\|^2, \tag{3.3}$$

for a.e. $t \in (0,T)$. From (3.3) and (P_1) we get

$$\begin{aligned} g'' &= (u'-v', u-v) + (Qu-Qv, u-v) + \|u'-v'\|^2 \\ &\ge g' + \|u'-v'\|^2 \quad \text{a.e. } t \in (0,T). \end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned} (e^{-t}g)'' &= e^{-t}(g''-2g'+g) \\ &\ge e^{-t}(\|u'-v'\|^2 - g' + g) \\ &= e^{-t}\left(\|u'-v'\|^2 - (u'-v', u-v) + \frac{1}{2}\|u-v\|^2\right) \\ &\ge \frac{1}{2}e^{-t}\|u'-v'\|^2 \end{aligned} \tag{3.5}$$

for a.e. $t \in (0,T)$, hence $t \mapsto e^{-t}g(t)$ is a convex function. This yields

$$e^{-t}g(t) \le \max\{g(0), e^{-T}g(T)\} \quad \forall t \in [0,T],$$

i.e., (3.1) holds. In order to prove estimation (3.2), consider the function (as in [11]) $\beta_\delta(t) = \min\{t-\delta, T-\delta-t\}$ for a small $\delta > 0$. If we multiply (3.5) by β_δ and integrate over $[\delta, T-\delta]$, we obtain

$$\begin{aligned} \frac{1}{2}\int_\delta^{T-\delta} e^{-t}\beta_\delta(t)\|u'-v'\|^2 dt &\le \int_\delta^{T-\delta} \beta_\delta(t)(e^{-t}g)'' dt = -\int_\delta^{T-\delta} \beta_\delta'(t)(e^{-t}g)' dt \\ &= e^{-\delta}g(\delta) + e^{-T+\delta}g(T-\delta) - 2e^{-T/2}g(T/2) \\ &\le g(\delta) + e^{-T+\delta}g(T-\delta). \end{aligned}$$

Letting $\delta \to 0^+$ and applying Fatou's lemma yields

$$\frac{1}{2}\int_0^T \beta(t)\|u'-v'\|^2 dt \le e^T g(0) + g(T),$$

where $\beta(t) := \min\{t, T-t\}$. This inequality implies (3.2), so the proof of Claim 1 is complete. □

Claim 2. *Let $u \in W^{2,2}(0,T;H)$ be the solution of (P_1) with $u_0, u_1 \in D(A)$, and $f \in L^2(0,T;H)$. Let $u_\lambda \in W^{2,2}(0,T;H)$ be the solution of the problem*

$$-u_\lambda'' + u_\lambda' + A_\lambda u_\lambda = f - Bu, \quad 0 \le t \le T, \tag{3.6a}$$

$$u_\lambda(0) = u_0, \qquad u_\lambda(T) = u_1, \tag{3.6b}$$

where $\lambda > 0$ and A_λ denotes the Yosida approximation of A. The existence of u_λ follows by Theorem 3.1, where $A := 0$, $B := A_\lambda$, and $f(t) := f(t) - Bu(t)$. Then,

$$u_\lambda \to u, \qquad u_\lambda' \to u' \text{ in } C([0,T];H), \tag{3.7}$$

$$\text{and} \quad u_\lambda'' \to u'' \quad \text{weakly in } L^2(0,T;H) \text{ as } \lambda \to 0^+. \tag{3.8}$$

Proof of Claim 2. Define for $\lambda > 0$ and $t \in [0,T]$

$$\begin{cases} u^*(t) = \dfrac{T-t}{T}u_0 + \dfrac{t}{T}u_1, \\ v_\lambda(t) = u_\lambda(t) - u^*(t). \end{cases}$$

Obviously, v_λ satisfies the problem

$$\begin{cases} -v_\lambda'' + v_\lambda' + A_\lambda v_\lambda = f - Bu - A_\lambda u^* + \dfrac{1}{T}(u_0 - u_1), \quad 0 \le t \le T, \\ v_\lambda(0) = 0 = v_\lambda(T). \end{cases} \tag{3.9}$$

If we multiply (3.9) by v_λ and integrate over $[0,T]$, we obtain

$$-\int_0^T (v_\lambda'', v_\lambda)dt + \underbrace{\frac{1}{2}\int_0^T \frac{d}{dt}\|v_\lambda\|^2 dt}_{=0} + \underbrace{\int_0^T (A_\lambda v_\lambda, v_\lambda)dt}_{\ge 0}$$

$$= \int_0^T \left(f - Bu + \frac{1}{T}(u_0 - u_1) - A_\lambda u^*, v_\lambda\right) dt. \tag{3.10}$$

Since

$$\begin{aligned} \|A_\lambda u^*(t)\| &= \left\| \frac{T-t}{T}A_\lambda u_0 + \frac{t}{T}A_\lambda u_1 \right\| \\ &\le \frac{T-t}{T}\|Au_0\| + \frac{t}{T}\|Au_1\| \le \max\{\|Au_0\|, \|Au_1\|\} < \infty, \end{aligned} \tag{3.11}$$

for all $t \in [0,T]$, we derive from (3.10)

$$\int_0^T \|v_\lambda'\|^2 dt \le K\left(\int_0^T \|v_\lambda\|^2 dt\right)^{1/2}, \tag{3.12}$$

where $K > 0$ is a constant. On the other hand,

$$v_\lambda(t) = \int_0^t v_\lambda'(s)ds, \quad 0 \le t \le T. \tag{3.13}$$

Combining (3.12) and (3.13), we get

$$\{v_\lambda'; \lambda > 0\} \quad \text{is bounded in } L^2(0,T;H), \tag{3.14}$$

$$\{v_\lambda; \lambda > 0\} \quad \text{is bounded in } C([0,T];H). \tag{3.15}$$

Since

$$\begin{aligned} \frac{d}{dt}(v_\lambda', A_\lambda v_\lambda) &= (v_\lambda'', A_\lambda v_\lambda) + \underbrace{(v_\lambda', A_\lambda v_\lambda')}_{\ge 0} \\ &\ge (v_\lambda'', A_\lambda v_\lambda) \quad \text{for a.e. } t \in (0,T), \end{aligned}$$

we can write, by using (3.9),

$$0 \ge \int_0^T (v_\lambda'', A_\lambda v_\lambda)dt = \int_0^T (v_\lambda' + A_\lambda v_\lambda - f_\lambda, A_\lambda v_\lambda)dt,$$

where

$$f_\lambda(t) := f(t) - Bu(t) - A_\lambda u^*(t) + \frac{1}{T}(u_0 - u_1).$$

Therefore, by (3.11) and (3.14), one gets

$$\begin{aligned}\|A_\lambda v_\lambda\|_{L^2}^2 &\le \|v_\lambda'\|_{L^2}\,\|A_\lambda v_\lambda\|_{L^2} + \|f_\lambda\|_{L^2}\|A_\lambda v_\lambda\|_{L^2}\\ &\le K_1\|A_\lambda v_\lambda\|_{L^2},\end{aligned}$$

so both

$$\{u_\lambda''; \lambda > 0\} \quad \text{and} \quad \{A_\lambda u_\lambda; \lambda > 0\} \text{ are bounded in } L^2 := L^2(0, T; H). \tag{3.16}$$

For $\lambda, \mu > 0$ we have from (3.6)

$$-\int_0^T (u_\lambda'' - u_\mu'', u_\lambda - u_\mu)dt + \int_0^T (u_\lambda' - u_\mu', u_\lambda - u_\mu)dt + \int_0^T (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu)dt = 0,$$

which implies that

$$\begin{aligned}\int_0^T \|u_\lambda' - u_\mu'\|^2 dt &= -\int_0^T \underbrace{(A_\lambda u_\lambda - A_\mu u_\mu, J_\lambda u_\lambda - J_\mu u_\mu)}_{\ge 0}\,dt - \int_0^T (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu)dt\\ &\le K_2(\lambda + \mu),\end{aligned}$$

where $J_\lambda = (I + \lambda A)^{-1}$. This shows that $\{u_\lambda'; \lambda > 0\}$ is a Cauchy sequence in L^2, hence convergent in L^2 as $\lambda \to 0^+$. Since

$$\begin{aligned}\|u_\lambda(t) - u_\mu(t)\| &= \left\| \int_0^T (u_\lambda'(s) - u_\mu'(s))ds \right\|\\ &\le T^{1/2}\|u_\lambda' - u_\mu'\|_{L^2} \quad \forall t \in [0, T],\end{aligned}$$

$\{u_\lambda; \lambda > 0\}$ converges in $C([0, T]; H)$. Denote its limit by $\hat{u}$. Summarizing, we have $\hat{u} \in W^{2,2}(0, T; H)$ and

$$u_\lambda \to \hat{u} \quad \text{in } C([0, T]; H), \tag{3.17}$$

$$u_\lambda' \to \hat{u}' \quad \text{in } L^2(0, T; H), \tag{3.18}$$

$$u_\lambda'' \to \hat{u}'' \quad \text{weakly in } L^2(0, T; H), \text{ as } \lambda \to 0^+. \tag{3.19}$$

In fact, since by (3.19) the sequence $\{u_\lambda'; \lambda > 0\}$ is equicontinuous,

$$u_\lambda' \to \hat{u}' \quad \text{in } C([0, T]; H) \text{ as } \lambda \to 0^+. \tag{3.20}$$

It is easily seen that

$$J_\lambda u_\lambda \to \hat{u} \quad \text{in } C([0, T]; H). \tag{3.21}$$

Indeed, for all $t \in [0, T]$,

$$\begin{aligned}\|J_\lambda u_\lambda(t) - \hat{u}(t)\| &\le \|J_\lambda u_\lambda(t) - u_\lambda(t)\| + \|u_\lambda(t) - \hat{u}(t)\|\\ &= \lambda\|A_\lambda u_\lambda(t)\| + \|u_\lambda(t) - \hat{u}(t)\|\\ &\le K_3\lambda + \|u_\lambda(t) - \hat{u}(t)\| \quad \text{(by (3.16))},\end{aligned}$$

which confirms our assertion.

Using the above pieces of information on u_λ, we can pass to the limit as $\lambda \to 0^+$ in (3.6a) regarded as an equation in L^2 to obtain

$$-\hat{u}'' + \hat{u}' + A\hat{u} = f - Bu, \quad 0 \le t \le T. \tag{3.22a}$$

We also have (see (3.17) and (3.6b))

$$\hat{u}(0) = u_0, \qquad \hat{u}(T) = u_1. \tag{3.22b}$$

Now, from (P_1), (3.22a) and (3.22b) we can easily see that $\hat{u} \equiv u$. This completes the proof of Claim 2. □

Claim 3. *Let $u \in W^{2,2}(0, T; H)$ be the solution of (P_1) with $u_0, u_1 \in D(A)$ and $f \in L^2(0, T; H)$. Then, there exist constants C_1, $C_2 > 0$ such that*

$$\|u''\|_{L^2_{**}} \le C_1(\|f\|_{L^2} + \|u'\|_{L^2_*} + \|u\|_{C([0,T];H)}) + C_2, \tag{3.23}$$

where (as in [11]) $L^2_* := L^2(0, T; H; \beta(t)dt)$, $L^2_{**} := L^2(0, T; H; \beta^3(t)dt)$.

Proof of Claim 3. Consider again problem (3.6). From the obvious inequality

$$\frac{d}{dt}(u_\lambda', A_\lambda u_\lambda) = (u_\lambda'', A_\lambda u_\lambda) + (u_\lambda', A_\lambda u_\lambda') \geq (u_\lambda'', A_\lambda u_\lambda),$$

we derive by multiplication by β^3 and integration over $[0, T]$,

$$-3\int_0^T \beta^2\beta'(u_\lambda', A_\lambda u_\lambda)dt \geq \int_0^T \beta^3(u_\lambda' + A_\lambda u_\lambda - f + Bu, A_\lambda u_\lambda)dt.$$

It follows that

$$\|A_\lambda u_\lambda\|_{L^2_{**}}^2 \leq 3\|u_\lambda'\|_{L^2_*}\|A_\lambda u_\lambda\|_{L^2_{**}} + \|u_\lambda'\|_{L^2_{**}}\|A_\lambda u_\lambda\|_{L^2_{**}} + \|f\|_{L^2_{**}}\|A_\lambda u_\lambda\|_{L^2_{**}} + \|Bu\|_{L^2_{**}}\|A_\lambda u_\lambda\|_{L^2_{**}}$$
$$\leq K_4\|A_\lambda u_\lambda\|_{L^2_{**}}(\|f\|_{L^2} + \|u_\lambda'\|_{L^2_*} + \|u\|_{C([0,T];H)}) + K_5,$$

and so

$$\|A_\lambda u_\lambda\|_{L^2_{**}} \leq K_4(\|f\|_{L^2} + \|u_\lambda'\|_{L^2_*} + \|u\|_{C([0,T];H)}) + K_6. \tag{3.24}$$

From (3.6a) we then derive

$$\|u_\lambda''\|_{L^2_{**}} \leq C_1(\|f\|_{L^2} + \|u_\lambda'\|_{L^2_*} + \|u\|_{C([0,T];H)}) + C_2. \tag{3.25}$$

By (3.7), (3.8) and (3.25) we obtain (3.23). □

Proof of Theorem 3.2 (*Continuation*)**.** Let us approximate $u_0, u_1 \in H$ by $u_{0n}, u_{1n} \in D(A)$, i.e.,

$$\|u_{0n} - u_0\| \to 0, \qquad \|u_{1n} - u_1\| \to 0 \quad \text{as } n \to \infty.$$

By Theorem 3.1, problem (P_1) with $u(0) = u_{0n}$, $u(T) = u_{1n}$ has a unique solution $u_n \in W^{2,2}(0, T; H)$. Now, estimates (3.1), (3.2) and (3.23) come into play showing that there exists a function $u \in C([0, T]; H) \cap W^{2,2}_{\text{loc}}(0, T; H)$, with $u' \in L^2_*$, $u'' \in L^2_{**}$, such that

$$u_n \to u \quad \text{in } C([0, T]; H), \tag{3.26}$$

$$u_n' \to u' \quad \text{in } L^2_*, \tag{3.27}$$

$$u_n'' \to u'' \quad \text{weakly in } L^2_{**}. \tag{3.28}$$

Regarding the equation

$$-u_n'' + u_n' + Au_n + Bu_n = f$$

as one in the space $L^2(\delta, T - \delta; H)$ for positive small δ's as we obtain by (3.26)–(3.28) that u satisfies for a.e. $t \in (0, T)$ the equation

$$-u'' + u' + Au + Bu = f.$$

In addition, by (3.26),

$$u(0) = \lim u_n(0) = u_0, \qquad u(T) = \lim u_n(T) = u_1.$$

The uniqueness of the solution follows by (3.1). □

Theorem 3.3. *Assume* (Hyp1) *and* (Hyp2) *hold and B is monotone. If* $u_0, u_1 \in D(A)$ *and* $f \in W^{1,2}(0, T; H)$*, then problem* (P_1) *has a unique solution* $u \in W^{2,2}(0, T; H) \cap W^{3,2}_{\text{loc}}(0, T; H)$*, with* $t^{3/2}(T - t)^{3/2}u''' \in L^2(0, T; H)$*. If* $u_0, u_1 \in H$ *and* $f \in W^{1,2}(0, T; H)$*, then* $u \in C([0, T]; H) \cap W^{3,2}_{\text{loc}}(0, T; H)$*, with* $t^{1/2}(T - t)^{1/2}u'$, $t^{3/2}(T - t)^{3/2}u''$, $t^{5/2}(T - t)^{5/2}u''' \in L^2(0, T; H)$.

Proof. Assume first $u_0, u_1 \in D(A)$ and $f \in W^{1,2}(0, T; H)$. By Theorem 3.1 problem (P_1) has a unique solution $u \in W^{2,2}(0, T; H)$. Consider again problem (3.6). We know that u_λ approximates u in the sense of (3.7) and (3.8). Note that $u_\lambda \in W^{3,2}(0, T; H)$ and

$$-u_\lambda''' + u_\lambda'' + A_\lambda u_\lambda' = (f - Bu)' \quad \text{for a.e. } t \in (0, T). \tag{3.29}$$

Now, if we multiply by β^3 the inequality

$$\frac{d}{dt}(u_\lambda'', A_\lambda u_\lambda') \geq (u_\lambda''', A_\lambda u_\lambda'), \tag{3.30}$$

and then integrate over $[0, T]$, we get

$$-3\int_0^T \beta^2\beta\prime(u''_\lambda, A_\lambda u'_\lambda)dt \geq \int_0^T \beta^3(u''_\lambda - f' + (Bu)' + A_\lambda u'_\lambda, A_\lambda u'_\lambda)dt.$$

As in the proof of Claim 3, we find

$$\|u'''_\lambda\|_{L^2_{**}} \leq \widetilde{C}_1(\|f'\|_{L^2} + \|u''_\lambda\|_{L^2_*} + \|u'\|_{C([0,T];H)}) + \widetilde{C}_2. \tag{3.31}$$

According to (3.7), (3.8) and (3.31) $u''' \in L^2_{**}$ and $\beta^{3/2}u'''_\lambda \to \beta^{3/2}u'''$ weakly in L^2, as $\lambda \to 0^+$.

Now assume $u_0, u_1 \in H$ and $f \in W^{1,2}(0, T; H)$. By Theorem 3.2 above, problem (P_1) has a unique solution $u \in C([0, T]; H) \cap W^{2,2}_{\rm loc}(0, T; H)$, with $u' \in L^2_*, u'' \in L^2_{**}$. So all we have to prove is that $u \in W^{3,2}_{\rm loc}(0, T; H)$ and $t^{5/2}(T-t)^{5/2}u''' \in L^2(0, T; H)$. As usual, we approximate u_0, u_1 by $u_{0n}, u_{1n} \in D(A)$, and denote by u_n the solution of the problem

$$\begin{cases} -u''_n + u'_n + Au_n + Bu_n = f, & 0 \leq t \leq T, \\ u_n(0) = u_{0n}, & u_n(T) = u_{1n}. \end{cases} \tag{3.32}$$

From the proof of Theorem 3.2, we know that (u_n) satisfies (3.26)–(3.28). Now, for an $n \in \mathbb{N}$ (arbitrary but fixed) and $\lambda > 0$, denote by $u_{n\lambda}$ the solution of the problem

$$\begin{cases} -u''_{n\lambda} + u'_{n\lambda} + A_\lambda u_{n\lambda} = f - Bu_n, & 0 \leq t \leq T, \\ u_{n\lambda}(0) = u_{0n}, & u_{n\lambda}(T) = u_{1n}. \end{cases} \tag{3.33}$$

We know from Claim 2 that

$$\begin{cases} u_{n\lambda} \to u_n, \quad u'_{n\lambda} \to u'_n \quad \text{in } C([0, T]; H) \quad \text{and} \\ u''_{n\lambda} \to u''_n \quad \text{weakly in } L^2(0, T; H) \text{ as } \lambda \to 0^+. \end{cases} \tag{3.34}$$

Obviously, $u_{n\lambda} \in W^{3,2}(0, T; H)$ and satisfies the equation

$$-u'''_{n\lambda} + u''_{n\lambda} + A_\lambda u'_{n\lambda} = (f - Bu_n)' \quad \text{for a.e. } t \in (0, T). \tag{3.35}$$

Let us multiply by β^5 the inequality

$$\frac{d}{dt}(u''_{n\lambda}, A_\lambda u'_{n\lambda}) \geq (u'''_{n\lambda}, A_\lambda u'_{n\lambda}),$$

and integrate over $[0, T]$. Thus we derive the estimate

$$\left(\int_0^T \beta^5\,\|u'''_{n\lambda}\|^2\right)^{1/2} \leq \widehat{C}_1(\|u''_{n\lambda}\|_{L^2_{**}} + \|u'_n\|_{L^2_*} + \|f'\|_{L^2}) + \widehat{C}_2. \tag{3.36}$$

As in the proof of Claim 3, we obtain from (3.33) an estimate similar to (3.25)

$$\|u''_{n\lambda}\|_{L^2_{**}} \leq C_1(\|f\|_{L^2} + \|u'_{n\lambda}\|_{L^2_*} + \|u_n\|_{C([0,T];H)}) + C_2. \tag{3.37}$$

Using (3.37) in (3.36) we obtain

$$\left(\int_0^T \beta^5\|u'''_{n\lambda}\|^2\right)^{1/2} \leq D_1(\|f\|_{L^2} + \|f'\|_{L^2} + \|u'_{n\lambda}\|_{L^2_*} + \|u'_n\|_{L^2_*} + \|u_n\|_{C([0,T];H)}) + D_2, \tag{3.38}$$

where D_1, D_2 are positive constants. Since $u'_{n\lambda} \to u'_n$ in $C([0, T]; H)$ as $\lambda \to 0^+$ (see (3.34)), we deduce from (3.38) that $\{u'''_{n\lambda}; \lambda > 0\}$ is bounded in $L^2(0, T; H; \beta^5(t)dt)$, so $\beta^{5/2}u'''_{n\lambda} \to \beta^{5/2}u'''_n$ weakly in $L^2(0, T; H)$. Letting $\lambda \to 0^+$ in (3.38) we get

$$\|\beta^{5/2}u'''_n\|_{L^2} \leq D_1(\|f\|_{L^2} + \|f'\|_{L^2} + 2\|u'_n\|_{L^2_*} + \|u_n\|_{C([0,T];H)}) + D_2. \tag{3.39}$$

Since (u_n) satisfies (3.26) and (3.27), it follows from (3.39) that $\beta^{5/2}u''' \in L^2(0, T; H)$. Thus the proof of Theorem 3.3 is complete. □

Remark 3.4. By using the above method, we can obtain higher regularity for u under appropriate regularity assumptions on f and B.

4. Asymptotic expansion of order zero for (P_1^ε)

From the above analysis, we see that the solutions of problem (P_1^ε) are more regular than those of (P_0), as expected. For example, if $u_0, u_1 \in H$ and $f \in L^2(0,T;H)$, then the solution of (P_0) belongs to $C([0,T];H)$, while the solution of (P_1^ε) belongs to $C([0,T];H) \cap W^{2,2}_{loc}(0,T;H)$. Now if $u_0, u_1 \in D(A)$ and $f \in W^{1,2}(0,T;H)$, then the solution of (P_0) belongs to $C^1([0,T];H)$, while that of (P_1^ε) belongs to $W^{2,2}(0,T;H) \cap W^{3,2}_{loc}(0,T;H)$ (cf. Theorems 2.10 and 3.3).

We also expect that the solution of (P_1^ε) approximates the solution of (P_0) as $\varepsilon \to 0^+$. We will show in what follows that this is indeed the case under suitable conditions on the data. However, a boundary layer occurs near $t = T$ and so the solution u_ε of (P_1^ε) must be corrected by adding a boundary layer function in order to obtain a good approximation for the solutions of (P_0).

According to the previous results related to the particular cases of (P_1^ε) ([7], p. 211) the following expansion is expected to hold

$$u_\varepsilon(t) = u(t) + i(\tau) + r_\varepsilon(t), \quad 0 \le t \le T, \tag{4.1}$$

where $\tau := \frac{T-t}{\varepsilon}$ is the stretched (fast) variable, $u = u(t)$ is the solution of the reduced problem (P_0), $i = i(\tau)$ is the boundary layer function, and $r_\varepsilon = r_\varepsilon(t)$ is the remainder (of order zero).

Assuming that all functions involved in (4.1) are smooth enough, we can identify these functions by heuristic arguments. So if we use (4.1) in (E_ε), and identify the coefficients of $\varepsilon^{-1}, \varepsilon^0$, we get

$$\frac{d^2 i}{d\tau^2} + \frac{di}{d\tau} = 0, \quad \tau > 0 \quad \text{with } i(0) = u_1 - u(T), \tag{4.2}$$

u satisfies (P_0), and r_ε satisfies

$$\begin{cases} -\varepsilon(u + r_\varepsilon)'' + r_\varepsilon' + Ar_\varepsilon + B(u_\varepsilon) = B(u) - Ai, \\ r_\varepsilon(0) = -i(T/\varepsilon), \qquad r_\varepsilon(T) = 0. \end{cases} \tag{R_ε}$$

From (4.2) we get (note that $i(\infty) = 0$)

$$i(\tau) = (u_1 - u(T))e^{-\tau}. \tag{4.3}$$

Condition $i(\infty) = 0$ should be read as: i is negligible away from the boundary layer. For more details on the heuristic procedure to determine asymptotic expansions, see, e.g., [7]. In what follows we validate expansion (4.1).

Theorem 4.1. *Assume that* (Hyp1) *and* (Hyp2) *hold, B is a monotone operator, A is strongly positive, $u_0, u_1 \in D(A)$ and $f \in W^{1,1}(0,T;H)$. Then, for every $\varepsilon > 0$, the solution u_ε of problem (P_1^ε) admits the asymptotic expansion (4.1), where u is the solution of problem (P_0), $i = i(\tau)$ is the boundary layer function defined by (4.3), and the remainder $r_\varepsilon = r_\varepsilon(t)$ satisfies problem (R_ε) and the following estimate*

$$\|r_\varepsilon\|_{C([0,T];H)} = \mathcal{O}(\varepsilon^{1/4}). \tag{4.4}$$

Proof. By Theorems 2.10 and 3.1, we have

$$r_\varepsilon = u_\varepsilon - u - i \in C^1([0,T];H), \quad \text{and} \tag{4.5}$$

$$u + r_\varepsilon = u_\varepsilon - i \in W^{2,2}(0,T;H). \tag{4.6}$$

Note that $u(T) \in D(A)$, so $i(\tau) \in D(A)$ for all $\tau \ge 0$. It is easy to check that r_ε satisfies problem (R_ε).

In order to homogenize the boundary condition for r_ε, we set

$$\bar{r}_\varepsilon(t) = r_\varepsilon(t) + \alpha_\varepsilon(t), \quad 0 \le t \le T, \tag{4.7}$$

where

$$\alpha_\varepsilon(t) = (1 - t/T)i(T/\varepsilon). \tag{4.8}$$

Obviously, $\bar{r}_\varepsilon$ satisfies the problem

$$-\varepsilon(u + \bar{r}_\varepsilon)'' + \bar{r}_\varepsilon' + A\bar{r}_\varepsilon + Bu_\varepsilon = h_\varepsilon + Bu, \qquad 0 \le t \le T, \tag{4.9a}$$

$$\bar{r}_\varepsilon(0) = 0, \qquad \bar{r}_\varepsilon(T) = 0, \tag{4.9b}$$

where

$$h_\varepsilon(t) := -i(T/\varepsilon) + A\alpha_\varepsilon(t) - Ai(\tau). \tag{4.10}$$

Multiplying (4.9a) by $\bar{r}_\varepsilon$ and integrating over $[0, T]$, we obtain

$$\varepsilon \int_0^T ((u+\bar{r}_\varepsilon)', \bar{r}_\varepsilon')dt + \frac{1}{2}\int_0^T \frac{d}{dt}\|\bar{r}_\varepsilon\|^2 dt + \int_0^T (A\bar{r}_\varepsilon, \bar{r}_\varepsilon)dt + \int_0^T (Bu_\varepsilon, \bar{r}_\varepsilon)dt$$
$$= \int_0^T (h_\varepsilon, \bar{r}_\varepsilon)dt + \int_0^T (Bu, \bar{r}_\varepsilon)dt. \tag{4.11}$$

Since B is monotone, we derive from (4.11)

$$\varepsilon \int_0^T \|\bar{r}_\varepsilon'\|^2 dt + \int_0^T (A\bar{r}_\varepsilon, \bar{r}_\varepsilon)dt \le \int_0^T (Bu - B(u-\alpha_\varepsilon + i), \bar{r}_\varepsilon)dt + \varepsilon\|u'\|_{L^2}\|\bar{r}_\varepsilon'\|_{L^2} + \|h_\varepsilon\|_{L^2}\,\|\bar{r}_\varepsilon\|_{L^2}. \tag{4.12}$$

Note that

$$\begin{aligned} &\|i(T/\varepsilon)\| = \mathcal{O}(\varepsilon^j) \quad \forall j \ge 1, \qquad \|i\|_{L^2} = \mathcal{O}(\varepsilon^{1/2}),\\ &\|A\,i\|_{L^2} \;=\; \|e^{-\tau}A(u_1 - u(T))\|_{L^2}\\ &\qquad\quad \;=\; \|A(u_1 - u(T))\|_{L^2}\,\|e^{-\tau}\|_{L^2(0,T)} = \mathcal{O}(\varepsilon^{1/2}). \end{aligned}$$

We estimate

$$\|B(u+i-\alpha_\varepsilon) - Bu\|_{L^2} \le C\|i-\alpha_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^{1/2}). \tag{4.13}$$

From (4.12) and (4.13) we obtain

$$\varepsilon\|\bar{r}_\varepsilon'\|_{L^2}^2 + c\|\bar{r}_\varepsilon\|_{L^2}^2 = \mathcal{O}(\varepsilon^{1/2})\|\bar{r}_\varepsilon\|_{L^2} + \|u'\|_{L^2}\,\|\bar{r}_\varepsilon'\|_{L^2}\varepsilon,$$

since A is strongly positive with a constant $c > 0$. This estimate implies (4.4). □

Comments

1. The results presented in this paper cover many specific problems in PDEs, in particular the example mentioned in Section 1. For this particular example, one can obtain, in addition to (4.4) (which reads in this case $\|r_\varepsilon\|_{C([0,T];L^2(\Omega))} = \mathcal{O}(\varepsilon^{1/4})$) the following estimate

$$\|u_\varepsilon - u\|_{L^2(0,T;H_0^1(\Omega))} = \mathcal{O}(\varepsilon^{1/2}). \tag{4.14}$$

Indeed, an inspection of the proof of Theorem 4.1, shows that

$$(A\bar{r}_\varepsilon, \bar{r}_\varepsilon)_{L^2(0,T;L^2(\Omega))} = \int_0^T \int_\Omega \nabla_x\bar{r}_\varepsilon . \nabla_x\bar{r}_\varepsilon \, dx\, dt = \mathcal{O}(\varepsilon),$$

which implies (4.14), since

$$\begin{aligned} \|i\|_{L^2(0,T;H_0^1(\Omega))} &= \|u_1 - u(T)\|_{H_0^1(\Omega)} \left(\int_0^T e^{-2\left(\frac{T-t}{\varepsilon}\right)} dt\right)^{1/2}\\ &= \mathcal{O}(\varepsilon^{1/2}). \end{aligned}$$

So in this case the boundary layer function i can be included in the remainder term r_ε, i.e., i disappears from expansion (4.1). In other words, the boundary layer is not visible in $L^2(0, T; H_0^1(\Omega))$ and problem (P_1^ε) is regularly perturbed in this space (while it is singularly perturbed in $C([0, T]; L^2(\Omega))$).

2. In fact, Theorem 4.1 holds under weaker assumptions on B, more precisely it suffices to assume that

$$B\colon H \to H \quad \text{is monotone and Lipschitz on bounded sets.} \tag{4.15}$$

Obviously, if
(Hyp 2)′. B is Fréchet differentiable on H and $B'\colon H \to L(H)$ is bounded on bounded sets, then B is Lipschitz on bounded sets.
To argue, let us revisit the proof of Theorem 4.1. First, we used Theorem 3.1 and the second part of Theorem 2.10.
Theorem 3.1 is still true if we assume the weaker condition (4.15), since we need only the fact that $Q = A + B$ is maximal monotone.
The second part of Theorem 2.10 is also valid if B satisfies (4.15). Indeed, for $r > 0$, let us consider the operator $B_r = B \circ \phi_r$, where ϕ_r is the radial retraction,

$$\phi_r(x) = \begin{cases} x & \text{if } \|x\| \le r, \\ \dfrac{r}{\|x\|}x & \text{if } \|x\| \ge r. \end{cases}$$

Since ϕ_r is Lipschitz (see, e.g., [12], p. 55), it follows that B_r is Lipschitz on H (with a Lipschitz constant depending on r). If $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$, then problem (P_0) with B_r instead of B has a strong solution $u_r \in C^1([0, T]; H)$ (cf. Theorem 2.10). If we multiply by $u_r(t)$ the equation

$$u_r'(t) + Au_r(t) + Bu_r(t) - B0 = f(t) - B0$$

and take into account the monotonicity of A and B (A need not be strongly positive) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_r(t)\|^2 \leq (\|f(t)\| + \|B0\|)\|u_r(t)\|$$

which implies that

$$\|u_r\|_{C([0,T];H)} \leq M,$$

where M is a constant depending on $\|f\|_{C([0,T];H)}$ and $\|B0\|$ (and independent of r). Therefore, if we choose $r > M$, then u_r is a solution of (P_0). The rest of the proof of Theorem 4.1 works well for B satisfying (4.15). We just point out that in (4.13) the arguments of B belong to a ball in H whose radius depends on $\|u\|_{C([0,T];H)}$ but is independent of ε. So Theorem 4.1 actually covers a larger class of B's.

Further relaxations are also possible in other theorems. For example, Theorem 2.13 is valid under the weaker condition (Hyp 2)$'$. This condition is also sufficient to guarantee the conclusions of Theorem 3.2, as one can see by a careful inspection of the proof of Theorem 3.2.

3. It is expected that higher order asymptotic expansions hold for problem (P_1^ε) under appropriate additional assumptions on u_0, u_1, f and B. We will investigate this conjecture in a forthcoming paper.

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