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On the method of alternating resolvents

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ABSTRACT

The work of Hundal [H. Hundal, An alternating projection that does not converge in norm, Nonlinear Anal. 57 (1) (2004) 35–61] has revealed that the sequence generated by the method of alternating projections converges weakly, but not strongly in general. In this paper, we present several algorithms based on alternating resolvents of two maximal monotone operators, A and B, that can be used to approximate common zeros of A and B. In particular, we prove that the sequences generated by our algorithms converge strongly. A particular case of such algorithms enables one to approximate minimum values of certain convex functionals.

1. Introduction and preliminaries

Let H be a real Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. An operator $A : D(A) \subset H \to 2^H$ is called monotone if it satisfies the monotonicity property

$$\langle x - x', y - y'\rangle \geq 0, \quad \forall (x, y), (x', y') \in \mathrm{graph}(A).$$

Equivalently, A is monotone if its graph is a monotone subset of the product space $H \times H$. If there is no monotone operator A' whose graph properly contains the graph of A, then A is called a maximal monotone operator. For a maximal monotone operator A, the resolvent of A, defined by $J_\beta^A := (I + \beta A)^{-1}$, is well defined on the whole space H and is single valued for every $\beta > 0$. Most importantly, J_β^A is nonexpansive; that is, for every $x, y \in H$, the inequality $\|J_\beta^A x - J_\beta^A y\| \leq \|x - y\|$ holds. See, for example, [1] for details.

We will use the following notations: given a sequence $(x_n)_{n\in\mathbb{N}_0}$, $\mathbb{N}_0 = \{0, 1, \ldots\}$, (or (x_n) in short), and a point $x \in H$, $x_n \to x$ (respectively, $x_n \rightharpoonup x$) means that (x_n) converges strongly (respectively, weakly) to x. The weak ω-limit set of (x_n) will be denoted by $\omega_w((x_n))$. This set is defined as follows:

$$\omega_w((x_n)) = \left\{x \in H \mid x_{n_k} \rightharpoonup x \text{ for some subsequence } (x_{n_k})_{k\in\mathbb{N}_0} \text{ of } (x_n)_{n\in\mathbb{N}_0}\right\}.$$

The class of proper and convex functions from H into $(-\infty, \infty]$ will be denoted by $\Gamma(H)$. For any $\varphi \in \Gamma(H)$, the subdifferential (operator) $\partial\varphi : H \to H$ is defined by

$$\partial\varphi(x) = \{w \in H \mid \varphi(x) - \varphi(v) \leq \langle w, x - v\rangle \text{ for all } v \in H\}.$$

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A point $z \in H$ minimizes $\varphi \in \Gamma(H)$ iff $(z, 0) \in \partial\varphi$ (meaning that $z \in D(\partial\varphi)$ and $0 \in \partial\varphi(z)$). Recall that the subdifferential of a proper, lower semicontinuous and convex function is a maximal monotone operator. See, for example, [1, Theorem 1.12, p. 36] for details. Given a closed and convex subset C of H, the indicator function of C is a (proper) convex and lower semicontinuous function which gives the value zero at $x \in C$ and infinity outside C. Its subdifferential is the normal cone of C.

Now, let K_1 and K_2 be two nonempty, closed and convex sets in H with nonempty intersection, and consider the (convex feasibility) problem

$$\text{find an } x \in H \text{ such that } x \in K_1 \cap K_2. \tag{1}$$

The roots of this problem go as far back as the early 1930s, when von Neumann showed that, in the case when K_1 and K_2 are subspaces, the sequence of alternating projections

$$H \ni x_0 \mapsto x_1 = P_{K_1}x_0 \mapsto x_2 = P_{K_2}x_1 \mapsto x_3 = P_{K_1}x_2 \mapsto x_4 = P_{K_2}x_3 \mapsto \cdots$$

converges strongly to the point in the intersection of K_1 and K_2 which is the nearest to the starting point x_0. For the proof of this result, see, for example, [2] and the references therein. In 1965, Bregman [3] showed that, for two arbitrary closed and convex sets K_1 and K_2 with nonempty intersection, the sequence (x_n) generated by the method of alternating projections converges weakly to a point in $K_1 \cap K_2$. The question of whether or not (x_n) converges strongly remained open until recently, when Hundal [4] constructed an example in ℓ^2 showing that, for any starting point $x_0 \in \ell^2$, there exist a hyperplane K_1 and a cone K_2 such that $K_1 \cap K_2 = \{0\}$ and the sequence of alternating projections (x_n) converges weakly to zero, but not strongly; see also [5].

It should be noted that the projection operator coincides with the resolvent operator of a normal cone. Therefore, a natural way of extending the method of alternating projections is to consider two arbitrary maximal monotone operators, say A and B, instead of normal cones, in which case the method of alternating (or composition of) resolvents is defined as follows:

$$H \ni x_0 \mapsto x_1 = J^A_{\beta_1}x_0 \mapsto x_2 = J^B_{\gamma_1}x_1 \mapsto x_3 = J^A_{\beta_2}x_2 \mapsto x_4 = J^B_{\gamma_2}x_3 \mapsto \cdots,$$

where (β_n) and (γ_n) are sequences of positive real numbers. There are already many papers concerning this extension. In particular, for $\beta_n = \gamma_n = \lambda > 0$ for all $n \geq 1$, Bauschke et al. [6] showed that the sequence generated from this method converges weakly to a point of $\mathrm{Fix}\, J^A_\lambda J^B_\lambda$ – the fixed point set of the composition $J^A_\lambda J^B_\lambda$ – provided that this set is not empty. In this paper, we investigate the convergence properties of the sequence generated by the method of alternating resolvents defined above for the case when (β_n) and (γ_n) are not constant sequences. More precisely, we consider the inexact iterative method

$$x_{2n+1} = J^A_{\beta_n}(x_{2n} + e_n) \quad \text{for } n = 0, 1, \ldots, \tag{2}$$

$$x_{2n} = J^B_{\gamma_n}(x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \ldots, \tag{3}$$

where $x_0 \in H$ is a given starting point. We will prove that, under a summability condition on $(\|e_n\|)$ and $(\|e'_n\|)$, where (e_n) and (e'_n) are sequences of computational errors, the sequence generated by (2) and (3) is weakly convergent to a point in $A^{-1}(0) \cap B^{-1}(0)$ provided that this set is not empty, and that both (β_n) and (γ_n) are bounded from below away from zero. In this connection, see also [7], where an inexact resolvent iterative method for a single monotone operator is considered. In order to obtain strong convergence results for general maximal monotone operators A and B, a modification (following the idea from the case of a single maximal monotone operator (see [8–12])) of this method is carried out; see Section 3. With such a modification, the summability condition on the error sequences (e_n) and (e'_n) is also relaxed.

2. Preliminary lemmas

In this section, we present the necessary tools needed to prove our main results. The first lemma, which is due to Xu [8], is basic, yet very useful.

Lemma 1 (*see [8]*). *Let (s_n) be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - a_n)s_n + a_nb_n + c_n, \quad n \geq 0,$$

where (a_n), (b_n), and (c_n) satisfy the following conditions: (i) $(a_n) \subset [0, 1]$, *with* $\sum_{n=0}^\infty a_n = \infty$, (ii) $\limsup_{n\to\infty} b_n \leq 0$, *and* (iii) $c_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^\infty c_n < \infty$. *Then* $\lim_{n\to\infty} s_n = 0$.

Weak convergence results are proved with the aid of the following known lemma due to Opial.

Lemma 2 (*Z. Opial, (see, e.g., [1, p. 5])*). *Let F be a nonempty subset of H. Assume that the sequence (x_n) satisfies the following conditions:* (i) $\lim_{n\to\infty} \|x_n - q\| = \rho(q)$ *exists for all $q \in F$, and* (ii) *any weak cluster point of (x_n) belongs to F. Then, there exists a point $p \in F$ such that (x_n) converges weakly to p.*

Often, we shall use the following identity, the proof of which is well known and can easily be reproduced.

Lemma 3 (*Resolvent Identity*). *For any $\beta, \gamma > 0$, and $x \in H$, the identity*

$$J_\beta^A x = J_\gamma^A\left(\frac{\gamma}{\beta}x + \left(1 - \frac{\gamma}{\beta}\right)J_\beta^A x\right)$$

holds, where $A : D(A) \subset H \to 2^H$ is a maximal monotone operator.

Lemma 4 (*see [11]*). *For any sequence (b_n) of positive real numbers, the following conditions are equivalent:* (i) $\sum_{n=0}^\infty |b_{n+1} - b_n| < \infty$ *and* $0 < \liminf_{n\to\infty} b_n (= \lim_{n\to\infty} b_n)$, (ii) $\sum_{n=0}^\infty \frac{|b_{n+1}-b_n|}{b_n} < \infty$, *and* (iii) $\sum_{n=0}^\infty \frac{|b_{n+1}-b_n|}{b_{n+1}} < \infty$.

We conclude this section by showing that, whenever A (respectively, B) is strongly (and maximal) monotone and (β_n) (respectively, (γ_n)) is bounded from below away from zero, with the error sequences (e_n) and (e_n') being bounded, then the sequence (x_n) generated by (2) and (3) is bounded. In fact, we prove this result for coercive operators which strongly monotone operators are particular cases. Recall that an operator $A : D(A) \subset H \to 2^H$ is called coercive if it satisfies the following condition:

$$\lim_{\substack{\|\xi\|\to\infty,\\ (\xi,\eta)\in\mathrm{graph}(A)}} \frac{\langle \eta, \xi - v_0\rangle}{\|\xi\|} = \infty, \tag{4}$$

for some $v_0 \in H$. An operator A is called strongly monotone if there exists a positive constant c such that

$$\langle Au - Av, u - v\rangle \geq c\|u - v\|^2 \quad \forall u, v \in D(A).$$

Proposition 1. *Let $F := A^{-1}(0)\cap B^{-1}(0) \neq \emptyset$, where A and B are maximal monotone operators. Assume that the error sequences (e_n) and (e_n') are bounded. If either* (i) *A is coercive and (β_n) is bounded from below away from zero, or* (ii) *B is coercive and (γ_n) is bounded from below away from zero, then the sequence (x_n) generated by (2) and (3) is bounded.*

Proof. (The proof of this result is essentially borrowed from the proof of [1, Theorem 3.5, p. 152]). Fix $p \in F$, and let C^* be a positive constant such that

$$\|e_n\| + \|e_n'\| \leq C^* \quad \text{for all } n \geq 0.$$

Then we have from (3), and the fact that the resolvent operator is nonexpansive, that

$$\begin{aligned}\|x_{2n} - p\| &\leq \|x_{2n-1} - p + e_n'\| \\ &\leq \|x_{2n-1} - p\| + \|e_n'\| \quad \text{for all } n \geq 1.\end{aligned} \tag{5}$$

Similarly, from (2) and the above estimate, we have

$$\begin{aligned}\|x_{2n+1} - p\| &\leq \|x_{2n} - p\| + \|e_n\| \\ &\leq \|x_{2n-1} - p\| + \|e_n\| + \|e_n'\| \quad \text{for all } n \geq 1,\end{aligned}$$

which implies that

$$\|x_{2n+1}\| \leq \|x_{2n-1}\| + 2\|p\| + C^* \quad \text{for all } n \geq 1.$$

Now, let v_0 be the vector associated with the coercivity of A. Denote $C_1 := C^* + 2(\|p\| + \|v_0\|)$. Then, from (4), there exists a constant $K^* > 0$ such that

$$(\xi, \eta) \in A, \quad \|\xi\| > K^* \text{ implies } \frac{\langle \eta, \xi - v_0\rangle}{\|\xi - v_0\|} \geq \frac{C_1}{\varepsilon}, \tag{6}$$

where $\varepsilon > 0$ is the greatest lower bound of (β_n). If $\|x_{2n+1}\| \leq K^*$ for all $n \geq 0$, then this is clear. So we assume that there is an index k such that $\|x_{2k+1}\| > K^*$. Then, multiplying

$$x_{2k} - v_0 + e_k \in x_{2k+1} - v_0 + \beta_k A x_{2k+1}$$

by the unit vector $(x_{2k+1} - v_0)/\|x_{2k+1} - v_0\|$, and making use of (6), we get

$$\|x_{2k+1} - v_0\| + C_1 \leq \|x_{2k}\| + \|v_0\| + \|e_k\|,$$

which implies that

$$\|x_{2k+1}\| \leq \|x_{2k+1} - v_0\| + \|v_0\| \leq \|x_{2k}\| + 2\|v_0\| + \|e_k\| - C_1.$$

On the other hand, from (5), we derive

$$\|x_{2n}\| \leq \|x_{2n-1}\| + 2\|p\| + \|e_n'\| \quad \text{for all } n \geq 1.$$

Hence,

$$\begin{aligned}\|x_{2k+1}\| &\leq \|x_{2k-1}\| + 2\|v_0\| + 2\|p\| + \|e'_k\| + \|e_k\| - C_1 \\ &\leq \|x_{2k-1}\| + 2(\|v_0\| + \|p\|) + C^* - C_1 \\ &= \|x_{2k-1}\|.\end{aligned}$$

Therefore, we have, for each $n \geq 1$,

$$\|x_{2n+1}\| \leq \max\left\{K^* + C^* + 2\|p\|, \|x_{2n-1}\|\right\}. \tag{7}$$

Setting $\rho_n = \max\{K^* + C^* + 2\|p\|, \|x_{2n-1}\|\}$, we deduce from (7) that the sequence (ρ_n) is decreasing. Hence,

$$\|x_{2n+1}\| \leq \rho_n \leq \max\left\{K^* + C^* + 2\|p\|, \|x_1\|\right\} \quad \text{for all } n \geq 1,$$

showing that the subsequence (x_{2n+1}) of (x_n) is bounded, and so is the subsequence (x_{2n}) of (x_n) (see (5)). Hence, the sequence (x_n) itself is bounded. The proof of this result when B is coercive and (γ_n) is bounded from below away from zero is similar to the proof given above. □

3. Main results

In this section, we analyze the convergence properties of the sequences generated by the method of alternating resolvents. To motivate our discussion, we begin with the following (perhaps expected) weak convergence result.

Theorem 1. *Let A and B be maximal monotone operators with $F := A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Let (x_n) be the sequence generated by (2) and (3), where $\beta_n, \gamma_n \in (0, \infty)$. Assume that $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$. If both (β_n) and (γ_n) are bounded from below away from zero, then (x_n) converges weakly to a point in F for all $x_0 \in H$.*

Proof. It is worth pointing out that

$$x_{2n+1} + \beta_n A x_{2n+1} \ni x_{2n} + e_n \quad \text{for } n = 0, 1, \ldots, \tag{8}$$

and

$$x_{2n} + \gamma_n B x_{2n} \ni x_{2n-1} + e'_n \quad \text{for } n = 1, 2, \ldots, \tag{9}$$

are equivalent forms of Eqs. (2) and (3), respectively. Let us first note that, if the set F is nonempty, then the sequence $(\|x_n - p\|)$ is convergent for any $p \in F$; hence (x_n) is bounded. Indeed, for any $p \in F$, we have, from (3), that

$$\|x_{2n} - p\| \leq \|x_{2n-1} - p\| + \|e'_n\|. \tag{10}$$

Similarly, from (2), we derive

$$\|x_{2n+1} - p\| \leq \|x_{2n} - p\| + \|e_n\|,$$

which, together with (10), implies that

$$\|x_{2n+1} - p\| - \sum_{k=0}^{n}(\|e'_k\| + \|e_k\|) \leq \|x_{2n-1} - p\| - \sum_{k=0}^{n-1}(\|e'_k\| + \|e_k\|).$$

This shows that the sequence $(\|x_{2n+1} - p\|)$ is convergent. Similarly, $(\|x_{2n} - p\|)$ is convergent, with the same limit (see above). Consequently, the sequence $(\|x_n - p\|)$ is convergent, as claimed.

Now, subtracting x_{2n} from both sides of (8) (respectively, x_{2n-1} from both sides of (9)) and multiplying the resulting inclusion relation scalarly by $x_{2n+1} - p$ (respectively, by $x_{2n} - p$) for some $p \in F$, we get, upon the use of the monotonicity of A (respectively, of B),

$$\langle x_{2n+1} - x_{2n}, x_{2n+1} - p\rangle \leq \langle e_n, x_{2n+1} - p\rangle \quad \text{and} \quad \langle x_{2n} - x_{2n-1}, x_{2n} - p\rangle \leq \langle e'_n, x_{2n} - p\rangle,$$

respectively. Equivalently, we have, for some positive constant C,

$$\|x_{2n+1} - x_{2n}\|^2 + \|x_{2n+1} - p\|^2 - \|x_{2n} - p\|^2 \leq C\|e_n\|,$$

and

$$\|x_{2n} - x_{2n-1}\|^2 + \|x_{2n} - p\|^2 - \|x_{2n-1} - p\|^2 \leq C\|e'_n\|,$$

respectively. Adding these two last inequalities, and using the fact that $(\|x_n - p\|)$ is convergent, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{11}$$

Notice that, from (8), and the fact that (β_n) is bounded from below away from zero, we have

$$Ax_{2n+1} \ni \frac{x_{2n} - x_{2n+1} + e_n}{\beta_n} \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that $\omega_w((x_{2n+1})) \subset A^{-1}(0)$. Similarly, we derive from (9) the relation $\omega_w((x_{2n})) \subset B^{-1}(0)$. Therefore, from Lemma 2, there exists $v \in A^{-1}(0)$ and $w \in B^{-1}(0)$ such that $x_{2n+1} \rightharpoonup v$ and $x_{2n} \rightharpoonup w$.

We finally show that $v \equiv w$. Indeed, for all $z \in H$, we have

$$\langle v - w, z\rangle = \lim_{n\to\infty} \langle x_{2n+1} - w, z\rangle = \lim_{n\to\infty} \langle x_{2n+1} - x_{2n}, z\rangle + \lim_{n\to\infty} \langle x_{2n} - w, z\rangle = 0.$$

Consequently, $x_n \rightharpoonup v \in A^{-1}(0) \cap B^{-1}(0)$. □

Remark 1. If, in addition, either A or B is strongly monotone, then we have strong convergence to the unique element of F. Assume without loss of generality that B is strongly monotone, with monotonicity constant c. Then, since (x_n) is bounded, we multiply scalarly by $x_{2n} - p$, and get

$$\langle x_{2n} - x_{2n-1}, x_{2n} - p\rangle + \gamma_n \langle Bx_{2n}, x_{2n} - p\rangle \le M\|e'_n\|,$$

for some $M > 0$, where p is the unique point in F. Using the strong monotonicity of B, we have

$$\|x_{2n} - x_{2n-1}\|^2 + \|x_{2n} - p\|^2 - \|x_{2n-1} - p\|^2 + 2c\gamma_n\|x_{2n} - p\|^2 \le 2M\|e'_n\|. \tag{12}$$

On the other hand, since

$$\|x_{2n+1} - x_{2n}\|^2 + \|x_{2n+1} - p\|^2 - \|x_{2n} - p\|^2 \le C\|e_n\|, \tag{13}$$

for some positive constant C, we have

$$\|x_{2n+1} - x_{2n}\|^2 + 2c\gamma_n\|x_{2n} - p\|^2 \le \|x_{2n-1} - p\|^2 - \|x_{2n+1} - p\|^2 + C\|e_n\| + 2M\|e'_n\|.$$

Summing this last inequality from $n = 1$ to $n = \infty$, and using the fact that γ_n is bounded from below away from zero, one derives $x_{2n} - p \to 0$ and $x_{2n+1} - x_{2n} \to 0$. Therefore, (x_n) converges strongly to p.

Remark 2. When $\gamma_n \to \infty$ (in the case when B is strongly monotone) or $\beta_n \to \infty$ (in the case when A is strongly monotone), norm convergence of the error sequences (e_n) to zero and boundedness of (e'_n) (respectively, norm convergence of (e'_n) to zero and boundedness of (e_n)) are enough to guarantee strong convergence of (x_n) to the unique point $p \in F$. Indeed, we have from (12) that

$$\|x_{2n} - p\|^2 \le \frac{1}{1 + 2c\gamma_n}(\|x_{2n-1} - p\|^2 + 2M\|e'_n\|) \le \frac{K}{1 + 2c\gamma_n},$$

for some $K > 0$, where the last inequality follows from the fact that the sequences (x_n) and (e'_n) are bounded. (The boundedness of (x_n) follows from Proposition 1.) Therefore, passing to the limit in the above estimate, we see that (x_{2n}) is strongly convergent to p. On the other hand, passing to the limit in (13), we also derive strong convergence of (x_{2n+1}) to p. Consequently, the whole sequence (x_n) is strongly convergent to p, as claimed. The other case is proved analogously.

Remark 3. In the case when (e_n) and (e'_n) are zero for every n, then the sequence (x_n) converges to p at least at a linear rate, whenever both (β_n) and (γ_n) are bounded from below away from zero. In the case when B is strongly monotone and $\gamma_n \to \infty$ (respectively, A is strongly monotone and $\beta_n \to \infty$), the rate of convergence is improved to superlinearity. These facts follow from the following two inequalities:

$$\frac{\|x_{2n} - p\|}{\|x_{2n-2} - p\|} \le \frac{1}{\sqrt{1 + 2c\gamma_n}}, \quad \text{and} \quad \frac{\|x_{2n+1} - p\|}{\|x_{2n-1} - p\|} \le \frac{1}{\sqrt{1 + 2c\gamma_n}},$$

which are a result of combining inequalities (12) and (13) with $e_n = 0 = e'_n$ for all $n \ge 0$.

Remark 4. Strong convergence can also be obtained if, in addition to the assumptions of Theorem 1, either one of the resolvents $J^A := (I + A)^{-1}$ or $J^B := (I + B)^{-1}$ is compact (i.e., it maps bounded sets to relatively compact sets). Indeed, if, for instance, the former resolvent is compact, then we may write (with the help of the resolvent identity) Eq. (2) as

$$x_{2n+1} = J^A z_n, \quad \text{where } z_n = \frac{1}{\beta_n}(x_{2n} + e_n) + \left(1 - \frac{1}{\beta_n}\right) x_{2n+1}.$$

Since (z_n) is bounded and J^A is compact, there is at least one strongly convergent subsequence, say (x_{2n_k+1}), of (x_{2n+1}). Let q be the limit of (x_{2n_k+1}). Then, from (11), we get that $x_{2n_k} \to q$. Therefore, a subsequence (x_{n_k}) of (x_n) converges strongly to q. We note that $q \in F$, since $\omega_w((x_n)) \subset F$. On the other hand, $\lim \|x_n - p\|$ exists for all $p \in F$, so this limit is zero for $p = q$; i.e., (x_n) converges strongly to q.

Remark 5. We have shown in Theorem 1 that if the sequences $(\|e_n\|)$ and $(\|e'_n\|)$ are summable, and the parameters β_n and γ_n are bounded from below away from zero, then the sequence generated by (2) and (3) converges weakly to a point in $F := A^{-1}(0) \cap B^{-1}(0)$, provided that this set is not empty. Note that weak convergence of (x_n) (with $\beta_n = \gamma_n = \lambda$ for some $\lambda > 0$) was also shown by Bauschke et al. [6], where the set F was replaced by a generally larger set Fix $J_\lambda^A J_\lambda^B$. However, in the particular case when A and B are specialized to be subdifferentials of indicator functions of closed and convex sets K_1 and K_2 (that is, normal cones of K_1 and K_2, respectively), then $p \in K_1 \cap K_2$ (equivalently, a point p belongs to the set F) if and only if p is a fixed point of the composition mapping $P_{K_2}P_{K_1}$. Indeed, for $A = \partial I_{K_1}$, then $p \in A^{-1}(0) \Leftrightarrow p = J_\beta^A p = P_{K_1}p$ for some $\beta > 0$. Similarly, for $B = \partial I_{K_2}$, we have $p \in B^{-1}(0) \Leftrightarrow p = P_{K_2}p$. Therefore, $p \in A^{-1}(0) \cap B^{-1}(0)$ implies that $p = P_{K_2}P_{K_1}p$. Conversely, suppose that there exists a point $p \in H$ such that $p = P_{K_2}P_{K_1}p$. Then, it is obvious that $p \in K_2$. It only remains to show that $p \in K_1$. For this purpose, we derive from the inequality-characterizing projections that

$$\langle p - P_{K_1}p, p - v\rangle \le 0 \quad \text{for all } v \in K_2, \quad \text{and} \quad \langle p - P_{K_1}p, w - P_{K_1}p\rangle \le 0 \quad \text{for all } w \in K_1.$$

Therefore, if $K_1 \cap K_2$ is not empty, then, taking any point $y \in K_1 \cap K_2$ in place of v and w in the above inequalities, we readily establish that $p = P_{K_1}p$. Hence, $p \in K_1$, as desired.

It is worth pointing out that the above remark shows that, if $K_1 \cap K_2 \neq \emptyset$, then the sets $K_1 \cap K_2$ and Fix $P_{K_2}P_{K_1}$ coincide. In the case when $K_1 \cap K_2 = \emptyset$, then any of the following three cases may occur: (a) Fix $P_{K_2}P_{K_1} = K_2$, or (b) Fix $P_{K_2}P_{K_1} = \emptyset$, or (c) $\emptyset \neq$ Fix $P_{K_2}P_{K_1} \subsetneq K_2$. Indeed, taking K_1 and K_2 to be two parallel (and distinct) lines in $\mathbb{R}^2$, one can easily check that, for any point $x \in K_2$, we have $P_{K_2}P_{K_1}x = x$, and the intersection of K_1 and K_2 is the empty set. This verifies case (a). An easy way to see the validity of case (b) is to consider the two closed and convex sets $K_1 = \{x \times y \in \mathbb{R}^2 \mid x > 0 \text{ with } xy \ge 1\}$ and $K_2 = \{x \times y \in \mathbb{R}^2 \mid x < 0 \text{ with } -xy \ge 1\}$. We leave it to the reader to verify the other case.

The above discussion shows that the set Fix $P_{K_2}P_{K_1}$ is in general larger than the set $K_1 \cap K_2$. We refer the interested reader to the following papers which are relevant to the discussion in Remark 5: [13, Lemma 2.1, p. 463], [2, Theorem 4.1, p. 384], [5, Corollary 4.6, p. 424], and [14, Remark on page 286].

One can easily prove that the set Fix $P_{K_2}P_{K_1}$ contains at least one element if, in addition to the convexity and closedness of K_1 and K_2, either one of K_1 or K_2 is bounded. We state this fact more formally in the next proposition.

Proposition 2. *Let K_1 and K_2 be two nonempty, disjoint, closed and convex subsets of a real Hilbert space H. Assume that either K_1 or K_2 is bounded. Then Fix $P_{K_2}P_{K_1} \neq \emptyset$.*

Proof. Assume that K_2 is bounded. We first observe that the composition map $P_{K_2}P_{K_1} : H \to K_2$ is nonexpansive. Since K_2 is nonempty, closed, convex and bounded, it follows from the well-known Browder fixed point theorem that the map $P_{K_2}P_{K_1}$ restricted to K_2 has a fixed point (in K_2). Thus Fix $P_{K_2}P_{K_1} \neq \emptyset$.

Note that Fix $P_{K_2}P_{K_1} \neq \emptyset$ if and only if Fix $P_{K_1}P_{K_2} \neq \emptyset$. Therefore, in the case when K_1 is bounded, we derive (using similar arguments as above) Fix $P_{K_1}P_{K_2} \neq \emptyset$. Hence, Fix $P_{K_2}P_{K_1} \neq \emptyset$, as desired. □

Since the sequence generated by the method of alternating resolvents (Eqs. (2) and (3)) is in general only weakly convergent, even in the case of the method of alternating projections (see [4]), we modify this method in order to enforce strong convergence. Thus we define the sequence (x_n) iteratively by

$$x_{2n+1} = J_{\beta_n}^A(\alpha_n u + (1-\alpha_n)x_{2n} + e_n) \quad \text{for } n = 0, 1, \ldots, \tag{14}$$

$$x_{2n} = J_{\gamma_n}^B(x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \ldots, \tag{15}$$

where the points $x_0, u \in H$ are given and $e'_0 = 0$ by convention. The idea is borrowed from the regularization method for the case of a single maximal monotone operator (see [10,9]). Besides producing strongly convergent sequences, such a modification works well in the case when the error sequences (e_n) and (e'_n) are not summable; see Remark 6 below. Note also that, from (14) and (15) with $A = \partial I_{K_1}$ for $K_1 = H$, we reobtain the old modified proximal point algorithm (introduced by Xu [8] and Kamimura and Takahashi [15]) for $x_n := x_{2n}$. Similarly, if $B = \partial I_{K_2}$, where $K_2 = H$, we reobtain the regularization method [9], which is in fact equivalent [16] to the old modified proximal point algorithm for $x_n := x_{2n+1}$. Moreover, from (14) and (15) with $A = \partial I_{K_1}$ and $B = \partial I_{K_2}$, where K_1 and K_2 are nonempty, convex and closed subsets of H, we recover the method of alternating projections. With our modification, these algorithms become strongly convergent (under suitable assumptions), as the following theorem shows.

Theorem 2. *Let A and B be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $x_0, u \in H$, let (x_n) be the sequence generated by (14) and (15), where $\alpha_n \in (0, 1)$ and $\beta_n, \gamma_n \in (0, \infty)$. Assume that* (i) *$\lim_{n\to\infty} \alpha_n = 0$ with $\sum_{n=0}^\infty \alpha_n = \infty$, and either $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$ or $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, and that* (ii) *both (β_n) and (γ_n) are bounded from below away from zero, with*

$$\sum_{n=0}^\infty |\beta_{n+1} - \beta_n| < \infty \quad \text{and} \quad \sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty.$$

If the error sequences (e_n) and (e'_n) satisfy any of the following conditions,

(a) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *and* $\sum_{n=1}^{\infty}\|e'_n\|<\infty$;
(b) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *and* $\|e'_n\|/\alpha_n\to 0$, *with* $\sum_{n=1}^{\infty}\|e'_{n+1}-e'_n\|<\infty$;
(c) $\|e_n\|/\alpha_n\to 0$ *and* $\|e'_n\|/\alpha_n\to 0$, *with* $\sum_{n=1}^{\infty}\|e'_{n+1}-e'_n\|<\infty$;
(d) $\|e_n\|/\alpha_n\to 0$ *and* $\sum_{n=1}^{\infty}\|e'_n\|<\infty$,

then (x_n) *converges strongly to the point in* F *which is the nearest to* u.

Proof. We will prove the theorem only when the error sequences (e_n) and (e'_n) satisfy either condition (c) or condition (d). The reader will find it easy to adapt the proof given below to the other two cases.

The first step is to show that (x_n) is bounded. Take any $p\in F$. Then, from (14), (15), and the nonexpansivity of the resolvent, we have

$$\begin{aligned}\|x_{2n+1}-p\| &\le \|\alpha_n(u-p+e_n/\alpha_n)+(1-\alpha_n)(x_{2n}-p)\|\\ &\le \alpha_n\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}\right\}+(1-\alpha_n)\|x_{2n}-p\|\\ &\le \alpha_n C+(1-\alpha_n)\|x_{2n-1}-p\|+\|e'_n\|,\end{aligned}\tag{16}$$

where C is a positive constant such that

$$\sup\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}\,\middle|\, n\ge 0\right\}\le C.$$

Such a constant exists because the sequence $(\|e_n\|/\alpha_n)$ is bounded. Applying induction to (16) yields

$$\|x_{2n+1}-p\|\le C\left[1-\prod_{k=0}^{n}(1-\alpha_k)\right]+\|x_0-p\|\prod_{k=0}^{n}(1-\alpha_k)+\sum_{k=1}^{n}\|e'_k\|.$$

Therefore, under condition (d), the subsequence (x_{2n+1}) of (x_n) is bounded, and so is the subsequence (x_{2n}). Consequently, (x_n) is bounded.

In the case when both $(\|e_n\|/\alpha_n)$ and $(\|e'_n\|/\alpha_n)$ are bounded, we have, from (14) and (15), that

$$\begin{aligned}\|x_{2n+1}-p\| &\le \alpha_n\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}\right\}+(1-\alpha_n)\|x_{2n}-p\|\\ &\le \alpha_n\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}\right\}+(1-\alpha_n)\left\{\|x_{2n-1}-p\|+\alpha_n\frac{\|e'_n\|}{\alpha_n}\right\}\\ &\le \alpha_n C^*+(1-\alpha_n)\|x_{2n-1}-p\|,\end{aligned}\tag{17}$$

where C^* is a positive constant such that

$$\sup\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}+\frac{\|e'_n\|}{\alpha_n}|n\ge 0\right\}\le C^*.$$

Applying induction to (17), we again derive that (x_n) is bounded.

Next, we show that the weak ω-limit set of (x_n) is contained in F; that is, $\omega_w((x_n))\subset F$. Using the resolvent identity, we have, from (14), that

$$\begin{aligned}\|x_{2n+3}-x_{2n+1}\| &= \Bigg\|J^A_{\beta_{n+1}}(\alpha_{n+1}u+(1-\alpha_{n+1})x_{2n+2}+e_{n+1})\\ &\qquad - J^A_{\beta_{n+1}}\left(\frac{\beta_{n+1}}{\beta_n}(\alpha_n u+(1-\alpha_n)x_{2n}+e_n)+\left(1-\frac{\beta_{n+1}}{\beta_n}\right)x_{2n+1}\right)\Bigg\|\\ &\le \Bigg\|\frac{\beta_{n+1}}{\beta_n}(1-\alpha_n)(x_{2n+2}-x_{2n})+\left(1-\frac{\beta_{n+1}}{\beta_n}\right)(x_{2n+2}-x_{2n+1})\\ &\qquad +\left(\alpha_{n+1}-\frac{\beta_{n+1}\alpha_n}{\beta_n}\right)\left(u-x_{2n+2}+\frac{e_{n+1}}{\alpha_{n+1}}\right)+\frac{\beta_{n+1}\alpha_n}{\beta_n}\left(\frac{e_{n+1}}{\alpha_{n+1}}-\frac{e_n}{\alpha_n}\right)\Bigg\|\\ &\le \frac{\beta_{n+1}}{\beta_n}(1-\alpha_n)\|x_{2n+2}-x_{2n}\|+\left|1-\frac{\beta_{n+1}}{\beta_n}\right|K\\ &\qquad +\left|\alpha_{n+1}-\frac{\beta_{n+1}\alpha_n}{\beta_n}\right|K+\frac{\beta_{n+1}\alpha_n}{\beta_n}\left\|\frac{e_{n+1}}{\alpha_{n+1}}-\frac{e_n}{\alpha_n}\right\|,\end{aligned}\tag{18}$$

where K is a positive constant such that

$$\sup\left\{\left\|u - x_{2n+2} + \frac{e_{n+1}}{\alpha_{n+1}}\right\| + \|x_{2n+2} - x_{2n+1}\| | n \geq 0\right\} \leq K.$$

We now estimate $\|x_{2n+2} - x_{2n}\|$ as follows:

$$\begin{aligned}\|x_{2n+2} - x_{2n}\| &= \left\|J^B_{\gamma_n}\left(\frac{\gamma_n}{\gamma_{n+1}}(x_{2n+1} + e'_{n+1}) + \left(1 - \frac{\gamma_n}{\gamma_{n+1}}\right)x_{2n+2}\right) - J^B_{\gamma_n}(x_{2n-1} + e'_n)\right\| \\ &\leq \left\|(x_{2n+1} - x_{2n-1}) + \left(1 - \frac{\gamma_n}{\gamma_{n+1}}\right)(x_{2n+2} - x_{2n+1} - e'_{n+1}) + (e'_{n+1} - e'_n)\right\| \\ &\leq \|x_{2n+1} - x_{2n-1}\| + \left|1 - \frac{\gamma_n}{\gamma_{n+1}}\right| M + \|e'_{n+1} - e'_n\|,\end{aligned}$$

for some positive constant M. Therefore,

$$\begin{aligned}\frac{\|x_{2n+3} - x_{2n+1}\|}{\beta_{n+1}} \leq{}& (1 - \alpha_n)\frac{\|x_{2n+1} - x_{2n-1}\|}{\beta_n} + \left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right| K + \left|1 - \frac{\gamma_n}{\gamma_{n+1}}\right| M' \\ &+ \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right| K + \frac{\alpha_n}{\beta_n}\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| + \frac{\|e'_{n+1} - e'_n\|}{\beta_n},\end{aligned}$$

for some positive constant M'. Hence, from Lemmas 1 and 4, we derive

$$\frac{\|x_{2n+1} - x_{2n-1}\|}{\beta_n} \to 0 \;\Leftrightarrow\; \|x_{2n+1} - x_{2n-1}\| \to 0.$$

In fact, $\|x_{2n+2} - x_{2n}\| \to 0$, as well. Now, multiplying the inclusion

$$x_{2n+1} - x_{2n+2} + \beta_n A x_{2n+1} \ni \alpha_n(u - x_{2n} + e_n/\alpha_n) + x_{2n} - x_{2n+2},$$

scalarly by $x_{2n+1} - p$ (where $p \in F$), and using the monotonicity of A, we get

$$\langle x_{2n+1} - x_{2n+2}, x_{2n+1} - p\rangle \leq \alpha_n K' + L\|x_{2n} - x_{2n+2}\|, \tag{19}$$

for some positive constants K' and L. Similarly, multiplying the inclusion

$$x_{2n+2} - x_{2n+1} + \gamma_{n+1} B x_{2n+2} \ni e'_{n+1}$$

scalarly by $x_{2n+2} - p$, and using the monotonicity of B, we arrive at

$$\langle x_{2n+2} - x_{2n+1}, x_{2n+2} - p\rangle \leq L'\|e'_{n+1}\|, \tag{20}$$

for some constant $L' > 0$. Adding the inequalities (19) and (20) and passing to the limit in the resulting inequality yields

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{21}$$

Therefore, passing to the limit in

$$A x_{2n+1} \ni \frac{\alpha_n(u - x_{2n}) + x_{2n} - x_{2n+1} + e_n}{\beta_n}, \tag{22}$$

we see that $\omega_w((x_{2n+1})) \subset A^{-1}(0)$. Similarly, we derive $\omega_w((x_{2n})) \subset B^{-1}(0)$. It then follows from these two inclusions and (21) that $\omega_w((x_n)) \subset F = A^{-1}(0) \cap B^{-1}(0)$. This suffices to deduce that

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle \leq 0. \tag{23}$$

We remark that the set F is closed and convex, being the intersection of two closed and convex sets; therefore, there exists a unique point $q \in F$ such that $q = P_F u$. From (14) and (15), we have, for $\|e_n\|/\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$,

$$\begin{aligned}\|x_{2n+1} - q\|^2 &\leq \|(1 - \alpha_n)(x_{2n} - q) + \alpha_n(u - q + e_n/\alpha_n)\|^2 \\ &= (1 - \alpha_n)^2\|x_{2n} - q\|^2 + \alpha_n^2\|u - q + e_n/\alpha_n\|^2 + 2\alpha_n(1 - \alpha_n)\langle u - q + e_n/\alpha_n, x_{2n} - q\rangle \\ &\leq (1 - \alpha_n)\|x_{2n-1} - q\|^2 + C'\|e'_n\| + \alpha_n\left[\alpha_n C' + 2(1 - \alpha_n)\left\langle u - q + \frac{e_n}{\alpha_n}, x_{2n} - q\right\rangle\right],\end{aligned}$$

where C' is a positive constant such that

$$\sup\left\{\left\|u-q+\frac{e_n}{\alpha_n}\right\|^2+\|e_n'\|+2\|x_{2n-1}-q\| \,|\, n\geq 1\right\}\leq C',$$

or

$$\|x_{2n+1}-q\|^2 \leq (1-\alpha_n)\|x_{2n-1}-q\|^2+\alpha_n\left[C'\left(\alpha_n+\frac{\|e_n'\|}{\alpha_n}\right)+2(1-\alpha_n)\left\langle u-q+\frac{e_n}{\alpha_n}, x_{2n}-q\right\rangle\right],$$

in the case when $\|e_n\|/\alpha_n \to 0$ and $\|e_n'\|/\alpha_n \to 0$. Hence, from Lemma 1, we have (in both cases) $\|x_{2n+1}-q\| \to 0$ as $n\to\infty$. Moreover, since $x_{2n+1}-x_{2n}\to 0$, we also have $\|x_{2n}-q\|\to 0$. Consequently, $x_n\to q$ as $n\to\infty$. □

Example 1. The sequences

$$\alpha_n=\begin{cases}\dfrac{2}{3n} & \text{if } n \text{ is even}\\ \dfrac{1}{2n} & \text{if } n \text{ is odd}\end{cases} \qquad \beta_n=\begin{cases}\dfrac{2(n+1)}{3n} & \text{if } n \text{ is odd}\\ \dfrac{n+1}{2n} & \text{if } n \text{ is even}\end{cases}$$

satisfy the conditions

$$\sum_{n=0}^{\infty}\left|\frac{\alpha_{n+1}}{\beta_{n+1}}-\frac{\alpha_n}{\beta_n}\right|<\infty \quad\text{and}\quad \lim_{n\to\infty}\left(\frac{\alpha_{n+1}}{\alpha_n}-\frac{\beta_{n+1}}{\beta_n}\right)=0,$$

but α_n defined above does not satisfy either $\lim_{n\to\infty}\alpha_{n+1}/\alpha_n=1$ or $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$.

The modification carried out in (14) and (15) is not the only possible one that can be made. Alternatively, we could have opted to modify (3) instead of (2). However, we do not carry out such a modification as it yields similar results to those obtained from (14) and (15). The other possibility is to modify both (2) and (3), in which case the sequence (x_n) is generated via the algorithm

$$x_{2n+1}=J^A_{\beta_n}(\alpha_n u+(1-\alpha_n)x_{2n}+e_n) \quad \text{for } n=0,1,\ldots, \tag{24}$$

$$x_{2n}=J^B_{\gamma_n}(\alpha_n u+(1-\alpha_n)x_{2n-1}+e_n') \quad \text{for } n=1,2,\ldots, \tag{25}$$

for given $x_0, u\in H$. A consideration of (24) and (25) instead of (14) and (15) enables one to dispense with the assumption $\sum_{n=1}^{\infty}\|e_{n+1}'-e_n'\|<\infty$ appearing in conditions (c) and (d) of Theorem 2.

Theorem 3. *Let A and B be maximal monotone operators with $A^{-1}(0)\cap B^{-1}(0) =: F \neq \emptyset$. Fix $x_0, u\in H$, and let (x_n) be the sequence generated by (24) and (25), where $\alpha_n\in(0,1)$ and $\beta_n,\gamma_n\in(0,\infty)$. Assume that* (i) *$\lim_{n\to\infty}\alpha_n=0$ with $\sum_{n=0}^{\infty}\alpha_n=\infty$, and either $\lim_{n\to\infty}\alpha_{n+1}/\alpha_n=1$ or $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, and that* (ii) *both (β_n) and (γ_n) are bounded from below away from zero, with*

$$\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty.$$

If the error sequences (e_n) and (e_n') satisfy any of the following conditions

(a) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *and* $\sum_{n=1}^{\infty}\|e_n'\|<\infty$;
(b) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ *and* $\|e_n'\|/\alpha_n\to 0$;
(c) $\|e_n\|/\alpha_n\to 0$ *and* $\|e_n'\|/\alpha_n\to 0$;
(d) $\|e_n\|/\alpha_n\to 0$ *and* $\sum_{n=1}^{\infty}\|e_n'\|<\infty$,

then (x_n) converges strongly to the point in F which is the nearest to u.

Proof. We prove the result only for the case when condition (c) is satisfied.

As before, we begin by showing that (x_n) is bounded. As we have already seen in the proof of the previous theorem,

$$\|x_{2n+1}-p\|\leq\alpha_n\left\{\|u-p\|+\frac{\|e_n\|}{\alpha_n}\right\}+(1-\alpha_n)\|x_{2n}-p\|.$$

Similarly, we have

$$\|x_{2n}-p\|\leq\alpha_n\left\{\|u-p\|+\frac{\|e_n'\|}{\alpha_n}\right\}+(1-\alpha_n)\|x_{2n-1}-p\|.$$

Therefore,

$$\|x_{2n+1}-p\| \le \alpha_n C^* + (1-\alpha_n)\|x_{2n-1}-p\|,$$

where C^* is a positive constant such that

$$\sup_{n\ge 0}\left\{2\|u-p\| + \frac{\|e_n\|}{\alpha_n} + \frac{\|e'_n\|}{\alpha_n}\right\} \le C^*.$$

The boundedness of (x_n) follows easily from the above inequality. Dividing (18) by $\beta_{n+1}\gamma_{n+1}$, and since (γ_n) is bounded from below away from zero, we have

$$\frac{\|x_{2n+3}-x_{2n+1}\|}{\beta_{n+1}\gamma_{n+1}} \le (1-\alpha_n)\frac{\|x_{2n+2}-x_{2n}\|}{\beta_n\gamma_{n+1}} + \left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right|K' + \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right|K' + \frac{\alpha_n}{\beta_n\gamma_{n+1}}\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\|, \tag{26}$$

for some $K' > 0$. On the other hand, a computation similar to that of (18) yields

$$\begin{aligned}\|x_{2n+2}-x_{2n}\| \le{}& \frac{\gamma_{n+1}}{\gamma_n}(1-\alpha_n)\|x_{2n+1}-x_{2n-1}\| + \left|1-\frac{\gamma_{n+1}}{\gamma_n}\right|K^* \\ &+ \left|\alpha_{n+1} - \frac{\gamma_{n+1}\alpha_n}{\gamma_n}\right|K^* + \frac{\gamma_{n+1}\alpha_n}{\gamma_n}\left\|\frac{e'_{n+1}}{\alpha_{n+1}} - \frac{e'_n}{\alpha_n}\right\|,\end{aligned} \tag{27}$$

for some positive constant K^*.

Now, dividing (27) by $\beta_n\gamma_{n+1}$, and since (β_n) is bounded from below away from zero, we have

$$\frac{\|x_{2n+2}-x_{2n}\|}{\beta_n\gamma_{n+1}} \le (1-\alpha_n)\frac{\|x_{2n+1}-x_{2n-1}\|}{\beta_n\gamma_n} + \left|\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right|L + \left|\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right|L + \frac{\alpha_n}{\beta_n\gamma_n}\left\|\frac{e'_{n+1}}{\alpha_{n+1}} - \frac{e'_n}{\alpha_n}\right\|,$$

for some positive constant L. This last inequality, together with (26), gives

$$\begin{aligned}\frac{\|x_{2n+3}-x_{2n+1}\|}{\beta_{n+1}\gamma_{n+1}} \le{}& (1-\alpha_n)\frac{\|x_{2n+1}-x_{2n-1}\|}{\beta_n\gamma_n} + \left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right|K' + \left|\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right|L \\ &+ \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right|K' + \left|\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right|L + \frac{\alpha_n}{\beta_n\gamma_{n+1}}\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| + \frac{\alpha_n}{\beta_n\gamma_n}\left\|\frac{e'_{n+1}}{\alpha_{n+1}} - \frac{e'_n}{\alpha_n}\right\|.\end{aligned}$$

Hence, from Lemma 1, we derive

$$\frac{\|x_{2n+1}-x_{2n-1}\|}{\beta_n\gamma_n} \to 0 \;\Leftrightarrow\; \|x_{2n+1}-x_{2n-1}\| \to 0,$$

where the equivalence is due to the fact that both (β_n) and (γ_n) are convergent.

Multiplying the inclusion

$$x_{2n+2} - x_{2n+1} + \gamma_{n+1}Bx_{2n+2} \ni \alpha_{n+1}(u - x_{2n+1} + e'_{n+1}/\alpha_{n+1})$$

scalarly by $x_{2n+2}-p$, and using the monotonicity of B, we arrive at

$$\langle x_{2n+2}-x_{2n+1}, x_{2n+2}-p\rangle \le \alpha_{n+1}L^*,$$

for some constant $L^* > 0$. We have already shown in (19) that

$$\langle x_{2n+1}-x_{2n+2}, x_{2n+1}-p\rangle \le \alpha_n K' + L\|x_{2n}-x_{2n+2}\|,$$

so, adding these two inequalities, and passing to the limit in the resulting inequality, we arrive at

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Therefore, as in (22), we derive the inclusions $\omega_w((x_{2n+1})) \subset A^{-1}(0)$ and $\omega_w((x_{2n})) \subset B^{-1}(0)$. Since $x_{n+1}-x_n \to 0$, these inclusions imply that $\omega_w((x_n)) \subset F$, and hence

$$\limsup_{n\to\infty}\langle u - P_F u, x_n - P_F u\rangle \le 0.$$

Finally, we show that $x_n \to P_F u$. From (24), we have

$$\begin{aligned}\|x_{2n+1}-P_F u\|^2 \le{}& \|(1-\alpha_n)(x_{2n}-P_F u) + \alpha_n(u - P_F u + e_n/\alpha_n)\|^2 \\ ={}& (1-\alpha_n)^2\|x_{2n}-P_F u\|^2 + \alpha_n^2\|u-P_F u + e_n/\alpha_n\|^2 \\ &+ 2\alpha_n(1-\alpha_n)\langle u - P_F u + e_n/\alpha_n, x_{2n}-P_F u\rangle.\end{aligned} \tag{28}$$

Similarly, we have, from (25), that

$$\|x_{2n} - P_F u\|^2 \le (1-\alpha_n)^2 \|x_{2n-1} - P_F u\|^2 + \alpha_n^2 \left\|u - P_F u + e_n'/\alpha_n\right\|^2 + 2\alpha_n(1-\alpha_n)\left\langle u - P_F u + e_n'/\alpha_n, x_{2n-1} - P_F u\right\rangle,$$

which, together with (28), gives

$$\|x_{2n+1} - P_F u\|^2 \le (1-\alpha_n)\|x_{2n-1} - P_F u\|^2 + \alpha_n b_n,$$

where

$$b_n = \alpha_n C_1 + 2(1-\alpha_n)\left[\left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{2n} - P_F u\right\rangle + \left\langle u - P_F u + \frac{e_n'}{\alpha_n}, x_{2n-1} - P_F u\right\rangle\right],$$

for some positive constant C_1. Therefore, from Lemma 1, we derive $x_{2n+1} \to P_F u$. Since $x_{n+1} - x_n \to 0$, we deduce strong convergence of (x_n) to $P_F u$. □

We will conclude this section by briefly explaining how the algorithms presented in this paper work under the error condition(s) $\|e_n\|/\alpha_n \to 0$ (and/or $\|e_n'\|/\alpha_n \to 0$) of Theorem 2 (Theorem 3). The condition $\sum_{n=0}^{\infty} \|e_n\| < \infty$ needs no further elaboration, as it has been widely used by several authors.

Remark 6. The nonsummability condition on the sequence of computational errors appearing in Theorem 2 was introduced by the authors in [16]. As explained in [16] (see also [11,12]), such a condition renders the algorithm in question applicable in approximating zeros of maximal monotone operators for every sequence of errors converging to zero in norm. In the current setting – where two nonlinear operators are involved – for sequences of summable errors, the algorithm works as expected; that is, one chooses the sequence (α_n) independent of the errors involved and executes the algorithm as usual. However, as soon as the sequence of computational errors associated with either one of the operators is norm convergent to zero and satisfies the condition $\sum_{n=0}^{\infty} \|e_n\| = \infty$, then an appropriate construction of the sequence of parameters (α_n) is carried out. Such a construction always depends on the error sequence (e_n), as it must meet the condition $\|e_n\|/\alpha_n \to 0$ for Theorem 2 to be applicable. The process of finding common zeros of two maximal monotone operators by means of iterative processes thus entails dividing the error sequences into two classes – the class of errors satisfying the condition $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and the ones that satisfy the condition $\sum_{n=0}^{\infty} \|e_n\| = \infty$ – and constructing an algorithm in accordance with the rules introduced and discussed in this section.

Remark 7. An extension of the algorithm defined by (24) and (25) is obtained by choosing different α_n in (24) and (25). This case requires separate analysis, and it will be discussed in a forthcoming paper.

4. The case when *A* and *B* are subdifferentials

Let $f, h : H \to (-\infty, +\infty]$ be two proper, convex and lower semicontinuous functions. For $A = \partial f$ and $B = \partial h$, we note that, if both β_n and γ_n are bounded below away from zero, and the sequences $(\|e_n\|)$ and $(\|e_n'\|)$ are summable, then the sequence (x_n) generated by (2) and (3) converges weakly to an element in $A^{-1}(0) \cap B^{-1}(0)$, provided that this intersection is nonempty. In addition, $\lim_{n\to\infty} f(x_{2n+1}) = \inf_{x \in H} f(x)$; thus the subsequence (x_{2n+1}) of (x_n) approximates minimum values of the convex functional $f : H \to (-\infty, +\infty]$. Similarly, the subsequence (x_{2n}) is used to approximate minimum values of the convex functional $h : H \to (-\infty, +\infty]$. Under the assumptions of Theorem 2 (respectively, Theorem 3), the sequence generated by (14) and (15) (respectively, by (24) and (25)) converges strongly to the common minimizer of f and g which is closest to u. This section further explores the case when A and B are subdifferentials of proper, convex and lower semicontinuous functions.

Theorem 4. *Assume that $f, h \in \Gamma(H)$ are lower semicontinuous, with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$, where $A = \partial f$ and $B = \partial h$. For any fixed $x_0 \in H$, let the sequence (x_n) be defined iteratively by (24) and (25), where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$. Assume also that the error sequences (e_n) and (e_n') satisfy any of the following conditions*

(a) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n'\| < \infty$;
(b) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $(\|e_n'\|/\alpha_n)$ *is bounded;*
(c) *both* $(\|e_n\|/\alpha_n)$ *and* $(\|e_n'\|/\alpha_n)$ *are bounded;*
(d) $(\|e_n\|/\alpha_n)$ *is bounded and* $\sum_{n=1}^{\infty} \|e_n'\| < \infty$.

If (β_n) and (γ_n) are both increasing and $\lim_{n\to\infty} \alpha_n = 0$, with $\sum_{n=0}^{\infty} \alpha_n \beta_n^{-1} < \infty$ and $\sum_{n=0}^{\infty} \alpha_n \gamma_n^{-1} < \infty$, then

$$\lim_{n\to\infty} f(x_{2n+1}) = \inf_{x\in H} f(x) \quad \text{and} \quad \lim_{n\to\infty} h(x_{2n}) = \inf_{y\in H} h(y). \tag{29}$$

The limits in (29) also hold true if both (β_n) and (γ_n) are bounded from below away from zero, (β_n) (or (γ_n)) is also bounded from above, and $\sum_{n=0}^{\infty} \alpha_n < \infty$. In this case, the sequence (x_n) is weakly convergent to a point in F.

Proof. We prove this result only for the case when condition (c) holds. The boundedness of (x_n) has already been shown in the proof of Theorem 3. Since

$$\beta_n \partial f(x_{2n+1}) \ni \alpha_n(u - x_{2n} + e_n/\alpha_n) + x_{2n} - x_{2n+1}, \tag{30}$$

for all $z \in D(f)$, we have, from the subdifferential inequality, that

$$\begin{aligned} 2\beta_n(f(x_{2n+1}) - f(z)) &\le 2\langle x_{2n} - x_{2n+1}, x_{2n+1} - z\rangle + 2\alpha_n\langle u - x_{2n} + e_n/\alpha_n, x_{2n+1} - z\rangle \\ &\le \|x_{2n} - z\|^2 - \|x_{2n+1} - z\|^2 - \|x_{2n} - x_{2n+1}\|^2 + \alpha_n M_z, \end{aligned}$$

where M_z is a positive constant such that

$$\sup\left\{\left\|u - x_{2n} + \frac{e_n}{\alpha_n}\right\|^2 + \|x_{2n+1} - z\|^2 | n \ge 0\right\} \le M_z.$$

In particular, for any $v \in F$,

$$\|x_{2n} - x_{2n+1}\|^2 \le \|x_{2n} - v\|^2 - \|x_{2n+1} - v\|^2 + \alpha_n M_v.$$

Similarly, starting with

$$\gamma_n \partial h(x_{2n}) \ni \alpha_n(u - x_{2n-1} + e'_n/\alpha_n) + x_{2n-1} - x_{2n}, \tag{31}$$

we derive

$$\|x_{2n} - x_{2n-1}\|^2 \le \|x_{2n-1} - v\|^2 - \|x_{2n} - v\|^2 + \alpha_n K_v,$$

for some positive constant K_v, which, together with the previous estimate, yields

$$\|x_{2n} - x_{2n+1}\|^2 \le \|x_{2n-1} - v\|^2 - \|x_{2n+1} - v\|^2 + \alpha_n C_v, \tag{32}$$

for some positive constant C_v. Dividing by β_n^2, and using the fact that (β_n) is increasing, we have, upon summing from $n = 1$ to $n = \infty$, that

$$\sum_{n=1}^{\infty}\left(\frac{\|x_{2n} - x_{2n+1}\|}{\beta_n}\right)^2 < \infty \Rightarrow \lim_{n\to\infty}\frac{\|x_{2n} - x_{2n+1}\|}{\beta_n} = 0. \tag{33}$$

Therefore, passing to the limit in

$$f(x_{2n+1}) - f(z) \le \left\langle\frac{x_{2n} - x_{2n+1}}{\beta_n}, x_{2n+1} - z\right\rangle + \frac{\alpha_n M_z}{2\beta_n},$$

we get

$$\limsup_{n\to\infty}(f(x_{2n+1}) - f(z)) \le 0 \quad \text{for all } z \in D(f).$$

This verifies the first part of (29). Similarly, passing to the limit in

$$h(x_{2n}) - h(w) \le \left\langle\frac{x_{2n-1} - x_{2n}}{\gamma_n}, x_{2n} - w\right\rangle + \frac{\alpha_n K_w}{2\gamma_n},$$

we get

$$\limsup_{n\to\infty}(h(x_{2n}) - h(w)) \le 0 \quad \text{for all } w \in D(h),$$

verifying the second part of (29).

To prove the last part of the theorem, assume that (β_n) is bounded, plus the other conditions specified in the statement. Then

$$\lim_{n\to\infty}\frac{\|x_{2n} - x_{2n+1}\|}{\beta_n} = 0 \Leftrightarrow \lim_{n\to\infty}\|x_{2n} - x_{2n+1}\| = 0. \tag{34}$$

Moreover, we have, from (32), that

$$\|x_{2n+1} - v\|^2 - C_v\sum_{k=0}^{n}\alpha_k \le \|x_{2n-1} - v\|^2 - C_v\sum_{k=0}^{n-1}\alpha_k,$$

showing that, for all $v \in F$, the sequence $(\|x_{2n+1} - v\|)$ is convergent. Hence, $(\|x_{2n} - v\|)$ is also convergent.

On the other hand, from (30) and (31), we derive $\omega_w((x_{2n+1})) \subset A^{-1}(0)$ and $\omega_w((x_{2n})) \subset B^{-1}(0)$, respectively. These two inclusions, together with (34), imply that $\omega_w((x_n)) \subset A^{-1}(0) \cap B^{-1}(0)$. Since the limit of the sequence $(\|x_n - v\|)$ exists for all $v \in F$, we have, via Lemma 2, that $x_n \rightharpoonup y$ for some $y \in F$. The proof is similar for the case when (γ_n) is bounded. $\square$

Remark 8. Recall [1, Prop. 2.7, p. 110] that, for a proper, convex and lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, the operator $(I + \partial\varphi)^{-1}$ is compact if and only if the set

$$K_{C,\varphi} := \{x \in H \,|\, \|x\| \le C, \text{ and } \varphi(x) \le C\}$$

is compact for every $C > 0$. Note that the set $K_{C,\varphi}$ is compact if and only if the set

$$M_{D,\varphi} := \{x \in H \,|\, \|x\|^2 + \varphi(x) \le D\}$$

is compact for every $D > 0$. Therefore, using similar arguments to those contained in Remark 4, one can show that, for $A = \partial f$ and $B = \partial h$, where $f, h \in \Gamma(H)$ are lower semicontinuous, the sequence (x_n) generated by (24) and (25) converges strongly to a common minimizer of f and h provided that both (β_n) and (γ_n) are bounded from below away from zero, either $M_{D,f}$ or $M_{D,h}$ is compact for all $D > 0$, $\sum_{n=0}^{\infty} \alpha_n < \infty$, and either (β_n) or (γ_n) is bounded from above.

5. Applications to variational inequalities

Let $K \subset H$ be a nonempty, closed and convex set, and let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator. Consider the problem

$$\text{Find } u \in K \cap D(A) \text{ such that there exists } z \in Au : \langle z, v - u\rangle \ge 0 \; \forall v \in K. \tag{35}$$

This is a variational inequality. It is easy to see that (35) is equivalent to the inclusion

$$0 \in Au + \partial I_K(u), \tag{36}$$

where I_K is the indicator function of K. If the set $F := K \cap A^{-1}(0)$ is assumed to be nonempty, then the method of alternating resolvents described above can be used to approximate points of F (which are solutions to (35)).

Example 2. Let Ω be an open and bounded subset of the Euclidean space $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Let $\beta : D(\beta) \subset \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone mapping; i.e., β is the subdifferential of a proper, convex, lower semicontinuous function $j : \mathbb{R} \to (-\infty, +\infty]$. Consider the boundary value problem

$$(BVP)\begin{cases} -\Delta u = f \quad \text{in } \Omega, \\ -\dfrac{\partial u}{\partial \nu} \in \beta(u) \quad \text{on } \partial\Omega, \\ u \ge 0 \quad \text{in } \Omega, \end{cases}$$

where $f \in L^2(\Omega)$ is a given function, and $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of u. Let H denote the space $L^2(\Omega)$ equipped with its usual scalar product and norm. Define $A : D(A) \subset H \to H$,

$$Av = -\Delta v - f,$$
$$D(A) = \left\{ v \in H^2(\Omega) \,|\, -\frac{\partial v}{\partial \nu} \in \beta(v), \text{ a.a } x \in \partial\Omega \right\},$$

where $H^2(\Omega)$ is the usual Sobolev space. Operator A is maximal monotone, being the subdifferential of the functional $\Phi : H \to (-\infty, +\infty]$

$$\Phi(v) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{2}|\nabla v|^2 - fv\right) dx + \int_{\partial\Omega} j(v)\, d\sigma, & \text{if } v \in H^1(\Omega),\ j(v) \in L^1(\partial\Omega), \\ +\infty, & \text{otherwise}, \end{cases}$$

which is proper, convex and lower semicontinuous. For details, see, for example, [17, Proposition 2.9, p. 62]. Let us assume that (BVP) has at least a solution $u \in H^2(\Omega)$. This means that $F := K \cap A^{-1}(0)$ is nonempty, where K denotes the cone $\{v \in H : v(x) \ge 0 \text{ for } a.a.\ x \in \Omega\}$. It is easy to find cases when this situation happens, and in general F is not a singleton. We consider a sequence (v_n) generated by (2) and (3), where A is the operator just defined above, and $B = \partial I_K$. Clearly, for all $\gamma > 0$, $J_\gamma^B v = P_K v = v^+ := \max\{v, 0\}$. For simplicity, we assume that $\beta_n = \gamma_n = 1$ for all n. Therefore, (2) and (3) become

$$v_{2n+1} = J^A(v_{2n} + e_n) \quad \text{for } n = 0, 1, \ldots, \tag{37}$$

$$v_{2n} = (v_{2n-1} + e'_n)^+ \quad \text{for } n = 1, 2, \ldots, \tag{38}$$

where v_0 is a given starting function. It is easy to check that J^A is a compact operator, so (v_n) is strongly convergent in $H = L^2(\Omega)$ to a point in $F := K \cap A^{-1}(0)$, whenever (e_n) and (e'_n) are summable in norm (see Theorem 1 and Remark 4). In fact, the sequence $y_n := v_{2n+1}$ generated by the difference equation

$$y_{n+1} = J^A(y_n^+ + e_n), \quad n = 0, 1, \ldots,$$

with y_0 a given starting function, converges strongly in $H^2(\Omega)$ to a solution of (BVP), under the summability condition on (e_n). Here we have incorporated the error e'_n into e_n; this is possible because P_K is a nonexpansive operator. In other words, (y_n) is a solution of the difference equation

$$\begin{cases} y_{n+1} - \Delta y_{n+1} = f + y_n^+ + e_n & \text{in } \Omega \\ -\dfrac{\partial y_{n+1}}{\partial \nu} \in \beta(y_{n+1}) & \text{on } \partial\Omega, \end{cases}$$

which allows us to derive strong convergence in $H^2(\Omega)$ for (y_n). Note that further convergence results for this particular example may be obtained from our abstract results above.

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