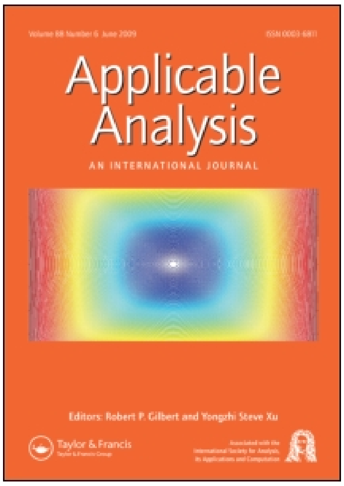

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Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions

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Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions

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We study a boundary value problem of the type $\sum_{i=1}^{N} \partial_{x_i}\big(a_i(x, \partial_{x_i} u)\big) = f(x, u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary and the functions $a_i(x, t)$ are of the type $|t|^{p_i(x)-2}t$ with $p_i(x) \in C(\overline{\Omega})$, $p_i(x) > 1$ $(i = 1, \ldots, N)$. Combining the mountain pass theorem of Ambrosetti and Rabinowitz and Ekeland's variational principle we show that under suitable conditions the problem has two non-trivial weak solutions.

Keywords: non-homogeneous differential operator; anisotropic Sobolev spaces; mountain pass theorem; Ekeland's variational principle

AMS Subject Classifications: 35D05; 35J60; 35J70; 58E05

1. Introduction

The goal of this article is to study the existence of solutions of a non-linear anisotropic problem of the type

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\big(a_i(x, \partial_{x_i} u)\big) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and the functions $a_i(x, t)$ are of the type $|t|^{p_i(x)-2}t$ with $p_i(x) \in C(\overline{\Omega})$, $p_i(x) > 1$ $(i = 1, \ldots, N)$. The function $f(x, t)$ in the right-hand side of Equation (1) satisfies some suitable conditions which will be specified later in this article.

In the particular case when $a_i(x, t) = |t|^{p(x)-2}t$ for all $i \in \{1, \ldots, N\}$, with $p(x) \in C(\overline{\Omega})$ and $\inf_\Omega p > 1$ the operator involved in (1) is the $p(\cdot)$-Laplace operator, i.e. $\Delta_{p(\cdot)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$. Moreover, our problem enables the presence of many other operators. We point out that we can also consider the particular case when $a_i(x, t) = (1 + t^2)^{(p(x)-2)/2}t$ for all $i \in \{1, \ldots, N\}$, with $p(x) \in C(\overline{\Omega})$ and $\inf_\Omega p > 1$. It is worth mentioning that problem (1) (considered in the isotropic

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case, i.e. when all the functions a_i are equal) can serve as a model for phenomena which arise from the study of electrorheological fluids [1–5], image processing [6] or the theory of elasticity [7].

Finally, we note that in the case when $a_i(x, t) = |t|^{m_i-2}t$ with $m_i > 1$ positive constants ($i = 1, \dots, N$), and $f(x, t) = \lambda|t|^{p-2}t$ with $p > 1$ and $\lambda > 0$, problem (1) was studied in [8] where some existence and non-existence results were proved.

2. Abstract framework

In this section, we recall some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. We will also introduce an adequate functional space where problems of type (1) can be studied. Such a space will be called an anisotropic variable exponent Sobolev space and it can be characterized as a functional space of Sobolev's type in which different space directions have different roles.

Set$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce *the variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \left\{u : u \text{ is a measurable real valued function such that} \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\right\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} \mathrm{d}x \le 1\right\},$$

which is a separable and reflexive Banach space. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_5$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder-type inequality

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2}$$

is valid.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

If u_n, $u \in L^{p(\cdot)}(\Omega)$ then the following implications hold

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}, \tag{3}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^-}, \tag{4}$$

$$|u_n - u|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u_n - u) \to 0, \tag{5}$$

since $p^+ < \infty$.

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)}.$$

We point out that the above norm is equivalent with the following norm

$$\|u\|_{p(\cdot)} = \sum_{i=1}^{N} \left|\partial_{x_i} u\right|_{p(\cdot)},$$

provided that $p(x) \geq 2$ for any $x \in \overline{\Omega}$ (see Proposition 1 at the end of this article). Hence $W_0^{1,p(\cdot)}(\Omega)$ is a separable, reflexive Banach space. Note that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, where $p^\star(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \geq N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous. For proofs, details and further results on variable exponent Lebesgue and Sobolev spaces we refer to the book of Musielak [9] and the papers of Kováčik and Rákosník [10], Edmunds et al. [11], Edmunds and Rakosnik [12,13], and Samko and Vakulov [14], while for applications of such kind of spaces to the study of partial differential equations we refer to [1,3–7,15–23].

Finally, we introduce a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ that will enable us to study problem (1) with sufficient accuracy. For this purpose, let us denote by $\vec{p} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\vec{p} = (p_1, \ldots, p_N)$. We define $W_0^{1,\vec{p}(\cdot)}(\Omega)$, the *anisotropic variable exponent Sobolev space*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\vec{p}(\cdot)} = \sum_{i=1}^{N} \left|\partial_{x_i} u\right|_{p_i(\cdot)}.$$

As it was pointed out in [20], $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

We also note that in the case when $p_i \in C_+(\overline{\Omega})$ are constant functions ($i = 1, \ldots, N$), the resulting anisotropic Sobolev space is denoted by $W_0^{1,\vec{p}}(\Omega)$, where $\vec{p}$ is the constant vector $(p_1, \ldots, p_N)$. The theory of such spaces was developed in [8,24–28].

On the other hand, in order to facilitate the manipulation of the space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ we introduce $\vec{P}_+, \vec{P}_- \in \mathbb{R}^N$ as

$$\vec{P}_+ = (p_1^+, \ldots, p_N^+), \quad \vec{P}_- = (p_1^-, \ldots, p_N^-),$$

and $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ as

$$P_+^+ = \max\{p_1^+, \ldots, p_N^+\}, \quad P_-^+ = \max\{p_1^-, \ldots, p_N^-\}, \quad P_-^- = \min\{p_1^-, \ldots, p_N^-\}.$$

Throughout this article we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{6}$$

and define $P^{\star}_{-} \in \mathbb{R}^{+}$ and $P_{-,\infty} \in \mathbb{R}^{+}$ by

$$P^{\star}_{-} = \frac{N}{\sum_{i=1}^{N} 1/p_i^{-} - 1}, \qquad P_{-,\infty} = \max\{P^{+}_{-}, P^{\star}_{-}\}.$$

We recall that if $s \in C_{+}(\overline{\Omega})$ satisfies $1 < s(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact (see [20, Theorem 1]).

3. The main result

For each $i \in \{1, \ldots, N\}$ assume that $a_i(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $A_i(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, is a primitive with respect to the variable t of $a_i(x,t)$, namely $A_i(x,t) = \int_0^t a_i(x,s)\,\mathrm{d}s$. Suppose that a_i and A_i satisfy the following hypotheses:

(A1) There exists a positive constant $c_{1,i}$ and a continuous function $p_i(x) : \overline{\Omega} \to [2, \infty)$ such that

$$|a_i(x,t)| \le c_{1,i}(1 + |t|^{p_i(x)-1}),$$

for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}$.
(A2) There exists $k_i > 0$ such that

$$A_i\left(x, \frac{t+s}{2}\right) \le \frac{1}{2} A_i(x,t) + \frac{1}{2} A_i(x,s) - k_i |t-s|^{p_i(x)}$$

for all $x \in \overline{\Omega}$ and $t, s \in \mathbb{R}$, where $p_i(x)$ is given in (A1).
(A3) The following inequalities hold true

$$|t|^{p_i(x)} \le a_i(x,t) \cdot t \le p_i(x)\, A_i(x,t),$$

for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, where $p_i(x)$ is given in (A1).

Examples

(1) Set $A_i(x,t) = \frac{1}{p_i(x)} |t|^{p_i(x)}$, $a_i(x,t) = |t|^{p_i(x)-2} t$, where $p_i(x) \ge 2$. Such a function contributes to Equation (7) with the term

$$\partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u).$$

(2) Set $A_i(x,t) = \frac{1}{p_i(x)}[(1+t^2)^{p_i(x)/2} - 1]$, $a_i(x,t) = (1+t^2)^{(p_i(x)-2)/2} t$, where $p_i(x) \ge 2$. Such a function contributes to Equation (7) with the term

$$\partial_{x_i}((1 + |\partial_{x_i} u|^2)^{(p_i(x)-2)/2} \partial_{x_i} u).$$

We will investigate problem (1) in the particular case

$$f(x,t) = \lambda(x) |t|^{q(x)} t,$$

where λ and q are two functions which satisfy certain properties that will be described below. Thus, problem (1) becomes

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\big(a_i(x, \partial_{x_i} u)\big) = \lambda(x)|u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

We assume that $\lambda : \Omega \to [0, \infty)$ satisfies the conditions:

(Λ_1) $\lambda \in L^\infty(\Omega)$;

(Λ_2) there exists an $x_0 \in \Omega$ and two positive constants r and R with $0 < r < R$ such that $\overline{B_R(x_0)} \subset \Omega$ and $\lambda(x) = 0$ for $x \in \overline{B_R(x_0) \setminus B_r(x_0)}$ while $\lambda(x) > 0$ for $x \in \Omega \setminus \overline{B_R(x_0) \setminus B_r(x_0)}$.

Moreover, we assume that the function $q : \overline{\Omega} \to [1, \infty)$ satisfies

(Q1) $q \in C(\overline{\Omega})$ and $1 \le q(x) < P_{-,\infty}$ for any $x \in \overline{\Omega}$;

(Q2) either $\max_{\overline{B_r(x_0)}} q < P_-^- \le P_+^+ < \min_{\overline{\Omega \setminus B_R(x_0)}} q$, or, $\max_{\overline{\Omega \setminus B_R(x_0)}} q < P_-^- \le P_+^+ < \min_{\overline{B_r(x_0)}} q$.

We state in what follows our main result regarding the existence of weak solutions to problem (7). By a *weak solution* to problem (7) we mean a function $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\int_\Omega \left\{ \sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi - \lambda(x) |u|^{q(x)-2} u \varphi \right\} \mathrm{d}x = 0$$

for all $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

THEOREM 1 *Assume that conditions* (A1)–(A3), (Λ_1)–(Λ_2) *and* (Q1)–(Q2) *are fulfilled. Then there exists a* $\lambda^\star > 0$ *such that problem* (7) *has at least two positive non-trivial weak solutions, provided that* $|\lambda|_{L^\infty(\Omega)} < \lambda^\star$.

4. Proof of the main result

From now on E denotes the anisotropic variable exponent Orlicz–Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Define the energetic functional associated with problem (7), $I : E \to \mathbb{R}$, by

$$I(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\, \mathrm{d}x - \int_\Omega \frac{\lambda(x)}{q(x)} |u|^{q(x)}\, \mathrm{d}x.$$

We first prove that functional I is well defined. Using hypothesis (A1) we get

$$\begin{aligned} A_i(x, t) &\le c_{1,i} \int_0^t (1 + |s|^{p_i(x)-1}) |s|\, \mathrm{d}s \\ &\le c_{1,i}|t| \frac{c_{1,i}}{p_i(x)} |t|^{p_i(x)} \\ &\le c_{1,i}|t| + \frac{c_{1,i}}{P_-^-} |t|^{p_i(x)}, \quad \forall x \in \overline{\Omega},\ t \in \mathbb{R},\ i \in \{1, \ldots, N\}. \end{aligned} \tag{8}$$

The above inequality and (A3) imply

$$0 \le \int_\Omega A_i(x, \partial_{x_i} u)\, \mathrm{d}x \le c_{1,i} \int_\Omega |\partial_{x_i} u|\, \mathrm{d}x + \frac{c_{1,i}}{P_-^-} \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, \mathrm{d}x, \quad \forall u \in E,\ i \in \{1, \ldots, N\}.$$

Using inequality (2) and relations (3) and (4) on the one hand, and relations (Λ_1), (Q1) and Theorem 1 in [20] on the other hand, we deduce that I is well defined on E.

Furthermore, similar arguments as those used in the proof of Proposition 3.1 in [3] assure that $I \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle I'(u), v\rangle = \int_\Omega \left\{ \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i} v \, \mathrm{d}x - \int_\Omega \lambda(x)|u|^{q(x)-2} uv \right\} \mathrm{d}x,$$

for all $u, v \in E$. Thus, we notice that we can seek weak solutions of problem (7) as critical points of the energetic functional I. Actually, we will show that for $|\lambda|_{L^\infty(\Omega)}$ small enough, the functional I possesses two critical points u_1 and $u_2 \in E$ such that $I(u_1) > 0$ and $I(u_2) < 0$. The former assertion will be obtained by using the mountain pass theorem while the latter will arise as a consequence of Ekeland's variational principle.

For our convenience, we define $J : E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u) \, \mathrm{d}x.$$

Lemma 1 *Assume that the sequence* (u_n) *converges weakly to* u *in* E and

$$\limsup_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u) \, \mathrm{d}x \le 0. \tag{9}$$

Then (u_n) *converges strongly to* u *in* E.

Proof The fact that u_n converges weakly to u in E implies that there exists $M > 0$ such that $\|u_n\|_{\vec{p}(\cdot)} \le M$ for all n. By relation (8) and inequalities (2)–(4) we deduce that $\{J(u_n)\}$ is bounded. Then, up to a subsequence, we deduce that $J(u_n) \to c$. But the functional J is weakly lower semi-continuous (because it is convex and of class C^1). Thus, we obtain

$$J(u) \le \liminf_{n\to\infty} J(u_n) = c. \tag{10}$$

On the other hand, since J is convex, we have

$$J(u) \ge J(u_n) + \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n) \cdot (\partial_{x_i} u - \partial_{x_i} u_n) \, \mathrm{d}x.$$

The above relation combined with (9) and (10) implies that $J(u) = c$.

Next, since (u_n) converges weakly to u in E, obviously, $(u_n + u)/2$ converges weakly to u in E. Using again the fact that J is weakly lower semi-continuous we have

$$c = J(u) \le \liminf_{n\to\infty} J\Big(\frac{u_n + u}{2}\Big). \tag{11}$$

Let us suppose by contradiction that (u_n) does not converge (strongly) to u in E. Then relation (5) implies that there exist $\epsilon > 0$ and a subsequence (u_{n_m}) of (u_n) such that

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i}(u_{n_m} - u)|^{p_i(x)} \, \mathrm{d}x \ge \epsilon, \quad \forall m. \tag{12}$$

Using condition (A2) we have

$$\begin{aligned}&\frac{1}{2}A_i(x,\partial_{x_i}u)+\frac{1}{2}A_i(x,\partial_{x_i}u_{n_m})-A_i\Big(x,\partial_{x_i}\frac{u+u_{n_m}}{2}\Big)\\&\quad\geq k_i|\nabla(u_{n_m}-u)|^{p_i(x)},\quad\forall\, x\in\overline{\Omega},\ i\in\{1,\ldots,N\}.\end{aligned}\tag{13}$$

Relations (12) and (13) yield

$$\frac{1}{2}J(u)+\frac{1}{2}J(u_{n_m})-J\Big(\frac{u+u_{n_m}}{2}\Big)\ \geq k\int_\Omega\sum_{i=1}^N|\partial_{x_i}(u_{n_m}-u)|^{p_i(x)}\,\mathrm{d}x\geq k\epsilon,$$

where $k=\min_{i\in\{1,\ldots,N\}}k_i$. Letting $m\to\infty$ in the above inequality we obtain

$$c-k\epsilon\geq\limsup_{m\to\infty}J\Big(\frac{u+u_{n_m}}{2}\Big)\,\mathrm{d}x$$

and this contradicts (11). It follows that u_n converges strongly to u in E and Lemma 6 is proved. ■

LEMMA 2

(i) *There exists* $\lambda^\star>0$ *such that provided* $|\lambda|_{L^\infty(\Omega)}<\lambda^\star$ *there exist* $\rho>0$ *and* $\alpha>0$ *for which* $I(u)\geq\alpha$, *for any* $u\in E$ *with* $\|u\|_{\vec{p}(\cdot)}=\rho$.
(ii) *There exists* $\psi\in E,\psi\neq 0$ *such that* $\lim_{t\to\infty}I(t\psi)=-\infty$.
(iii) *There exists* $\varphi\in E$, $\varphi\neq 0$ *such that* $I(t\varphi)<0$ *for* $t>0$ *small enough.*

Proof We will prove Lemma 2 assuming that the former condition of (Q2) holds true. A similar proof can be made if the latter condition holds true.

(i) Let us define $q_1:\overline{B_r(x_0)}\to[1,\infty)$, $q_1(x)=q(x)$ for any $x\in\overline{B_r(x_0)}$ and $q_2:\overline{\Omega\setminus B_R(x_0)}\to[1,\infty)$, $q_2(x)=q(x)$ for any $x\in\overline{\Omega\setminus B_R(x_0)}$. We also introduce the notation $q_1^-=\min_{x\in\overline{B_r(x_0)}}q_1(x)$, $q_1^+=\max_{x\in\overline{B_r(x_0)}}q_1(x)$, $q_2^-=\min_{x\in\overline{\Omega\setminus B_R(x_0)}}q_2(x), q_2^+=\max_{x\in\overline{\Omega\setminus B_R(x_0)}}q_2(x)$. Then by relations (Q1) and (Q2) we have

$$1\leq q_1^-\leq q_1^+<P_-^-\leq P_+^+<q_2^-\leq q_2^+<P_{-,\infty}.\tag{14}$$

Thus, we deduce that E is continuously embedded in $L^{q_i^\pm}(\Omega)$ for $i\in\{1,2\}$. It follows that there exists a positive constant C such that

$$\int_\Omega|u|^{q_i^\pm}\,\mathrm{d}x\leq C\|u\|_{\vec{p}(\cdot)}^{q_i^\pm},\quad\forall\, u\in E,\ i\in\{1,2\}.$$

It follows that there exist two positive constants D_1 and D_2 such that

$$\begin{aligned}\int_{B_r(x_0)}|u|^{q_1(x)}\,\mathrm{d}x&\leq\int_{B_r(x_0)}|u|^{q_1^-}\,\mathrm{d}x+\int_{B_r(x_0)}|u|^{q_1^+}\,\mathrm{d}x\\&\leq\int_\Omega|u|^{q_1^-}\,\mathrm{d}x+\int_\Omega|u|^{q_1^+}\,\mathrm{d}x\\&\leq D_1\Big(\|u\|_{\vec{p}(\cdot)}^{q_1^-}+\|u\|_{\vec{p}(\cdot)}^{q_1^+}\Big),\quad\forall\, u\in E,\end{aligned}\tag{15}$$

and

$$\begin{aligned}\int_{\Omega\setminus B_R(x_0)} |u|^{q_2(x)}\,\mathrm{d}x &\le \int_{\Omega\setminus B_R(x_0)} |u|^{q_2^-}\,\mathrm{d}x + \int_{\Omega\setminus B_R(x_0)} |u|^{q_2^+}\,\mathrm{d}x \\ &\le \int_\Omega |u|^{q_2^-}\,\mathrm{d}x + \int_\Omega |u|^{q_2^+}\,\mathrm{d}x \\ &\le D_2\left(\|u\|_{\vec{p}(\cdot)}^{q_2^-} + \|u\|_{\vec{p}(\cdot)}^{q_2^+}\right), \quad \forall\, u\in E. \end{aligned} \tag{16}$$

Next, we focus our attention on the case when $u\in E$ and $\|u\|_{\vec{p}(\cdot)}<1$. For such an element u we have $|\partial_{x_i}u|_{p_i(\cdot)}<1$ and, by relation (4), we obtain

$$\frac{\|u\|_{\vec{p}(\cdot)}^{P_+^+}}{N^{P_+^+-1}} = N\left(\frac{\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}}{N}\right)^{P_+^+} \le \sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{P_+^+} \le \sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{p_i^+} \le \sum_{i=1}^N \int_\Omega |\partial_{x_i}u|^{p_i(x)}\,\mathrm{d}x. \tag{17}$$

Let $D_0=\max\{D_1,D_2\}$. Relations (15)–(17) imply

$$\begin{aligned} I(u) &\ge \frac{1}{P_+^+N^{P_+^+-1}}\|u\|_{\vec{p}(\cdot)}^{P_+^+} - \int_{B_r(x_0)} \frac{\lambda(x)}{q(x)}|u|^{q(x)}\,\mathrm{d}x - \int_{\Omega\setminus B_R(x_0)} \frac{\lambda(x)}{q(x)}|u|^{q(x)}\,\mathrm{d}x \\ &\ge \frac{1}{P_+^+N^{P_+^+-1}}\|u\|_{\vec{p}(\cdot)}^{P_+^+} - \frac{|\lambda|_{L^\infty(\Omega)}\cdot D_0}{q^-}\left(\|u\|_{\vec{p}(\cdot)}^{q_1^-} + \|u\|_{\vec{p}(\cdot)}^{q_1^+} + \|u\|_{\vec{p}(\cdot)}^{q_2^-} + \|u\|_{\vec{p}(\cdot)}^{q_2^+}\right) \\ &\ge \left[\alpha_1\|u\|_{\vec{p}(\cdot)}^{P_+^+} - \alpha_2|\lambda|_{L^\infty(\Omega)}\left(\|u\|_{\vec{p}(\cdot)}^{q_1^-} + \|u\|_{\vec{p}(\cdot)}^{q_1^+}\right)\right] \\ &\quad + \left[\alpha_1\|u\|_{\vec{p}(\cdot)}^{P_+^+} - \alpha_2|\lambda|_{L^\infty(\Omega)}\left(\|u\|_{\vec{p}(\cdot)}^{q_2^-} + \|u\|_{\vec{p}(\cdot)}^{q_2^+}\right)\right] \end{aligned}$$

for any $u\in E$ with $\|u\|_{\vec{p}(\cdot)}<1$, where α_1 and α_2 are positive constants.
Since the function $g:[0,1]\to\mathbb{R}$ defined by

$$g(t) = \alpha_1 - \alpha_2 t^{q_2^+-P_+^+} - \alpha_2 t^{q_2^- - P_+^+}$$

is positive in a neighbourhood of the origin, it follows that there exists $\rho\in(0,1)$ such that $g(\rho)>0$. On the other hand, defining

$$\lambda^\star = \min\left\{1, \frac{\alpha_1}{2\alpha_2}\min\{\rho^{P_+^+-q_1^-}, \rho^{P_+^+-q_1^+}\}\right\} \tag{18}$$

we deduce that, provided $|\lambda|_{L^\infty(\Omega)}<\lambda^\star$, there exists $\alpha>0$ such that for any $u\in E$ with $\|u\|_{\vec{p}(\cdot)}=\rho$ we have

$$I(u)\ge\alpha.$$

(ii) Let $\psi\in C_0^\infty(\Omega)$, $\psi\ge 0$ and there exist $x_1\in\Omega\setminus B_R(x_0)$ and $\varepsilon>0$ such that for any $x\in B_\epsilon(x_1)\subset(\Omega\setminus B_R(x_0))$ we have $\psi(x)>0$. For any $t>1$ by (8) and relations (3) and (4) we find

$$\begin{aligned} I(t\psi) &= \int_\Omega \sum_{i=1}^N A_i(x,\partial_{x_i}t\psi)\,\mathrm{d}x - \int_\Omega \frac{\lambda(x)}{q(x)}|t\psi|^{q(x)}\,\mathrm{d}x \\ &\le \int_\Omega \sum_{i=1}^N c_{1,i}(|t|\,|\partial_{x_i}\psi| + |t|^{p_i(x)}\,|\partial_{x_i}\psi|^{p_i(x)})\,\mathrm{d}x - \int_{\Omega\setminus B_R(x_0)} \frac{\lambda(x)}{q(x)}|t\psi|^{q(x)}\,\mathrm{d}x \\ &\le t^{P_+^+}\int_\Omega \sum_{i=1}^N c_{1,i}(|\partial_{x_i}\psi| + |\partial_{x_i}\psi|^{p_i(x)})\,\mathrm{d}x - t^{q_2^-}\int_{\Omega\setminus B_R(x_0)} \frac{\lambda(x)}{q(x)}|\psi|^{q(x)}\,\mathrm{d}x. \end{aligned}$$

Since $P_+^+ < q_2^-$ we infer that $\lim_{t\to\infty} I(t\psi) = -\infty$.

(iii) Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and there exist $x_2 \in B_r(x_0)$ and $\epsilon > 0$ such that for any $x \in B_\epsilon(x_2) \subset B_r(x_0)$ we have $\varphi(x) > 0$. Letting $t < 1$ by (8) and relations (3) and (4) we find

$$
\begin{aligned}
I(t\varphi) &= \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} t\varphi)\, \mathrm{d}x - \int_\Omega \frac{\lambda(x)}{q(x)} |t\varphi|^{q(x)}\, \mathrm{d}x \\
&\leq \int_\Omega \sum_{i=1}^N c_{1,i}(|t|\,|\partial_{x_i}\varphi| + |t|^{p_i(x)}\,|\partial_{x_i}\varphi|^{p_i(x)})\, \mathrm{d}x - \int_{B_r(x_0)} \frac{\lambda(x)}{q(x)} |t\varphi|^{q(x)}\, \mathrm{d}x \\
&\leq t^{P_-^-} \int_\Omega \sum_{i=1}^N c_{1,i}(|\partial_{x_i}\varphi| + |\partial_{x_i}\varphi|^{p_i(x)})\, \mathrm{d}x - t^{q_1^+} \int_{B_r(x_0)} \frac{\lambda(x)}{q(x)} |\varphi|^{q(x)}\, \mathrm{d}x.
\end{aligned}
$$

Obviously,

$$
I(t\varphi) < 0,
$$

for any $t < \delta^{1/(P_-^- - q_1^+)}$, where

$$
0 < \delta < \min\left\{1, \frac{\int_{B_r(x_0)} \frac{\lambda(x)}{q(x)} |\varphi|^{q(x)}\, \mathrm{d}x}{\int_\Omega \sum_{i=1}^N c_{1,i}(|\partial_{x_i}\varphi| + |\partial_{x_i}\varphi|^{p_i(x)})\, \mathrm{d}x}\right\}.
$$

The proof of Lemma 2 is complete. ■

Proof of Theorem 1 By Lemma 2(i) and (ii) and the mountain pass theorem of Ambrosetti and Rabinowitz [29], we deduce the existence of a sequence $(u_n) \subset E$ such that

$$
I(u_n) \to \overline{c} > 0 \quad \text{and} \quad I'(u_n) \to 0 \ \ (\text{in } E^\star) \quad \text{as } n \to \infty. \tag{19}
$$

We prove that (u_n) is bounded in E. In order to do that, we assume by contradiction that passing if necessary to a subsequence, still denoted by (u_n), we have $\|u_n\|_{\vec{p}(\cdot)} \to \infty$ and that $\|u_n\|_{\vec{p}(\cdot)} > 1$ for all n.

Keeping the notations introduced in Lemma 2 we point out that relation (19) and the above considerations combined with inequality (A3) imply that for n large enough we have

$$
\begin{aligned}
1 + \overline{c} + \|u_n\|_{\vec{p}(\cdot)} &\geq I(u_n) - \frac{1}{q_2^-}\langle I'(u_n), u_n\rangle \\
&\geq \int_\Omega \sum_{i=1}^N \left(\frac{1}{p_i(x)} - \frac{1}{q_2^-}\right) |\partial_{x_i} u_n|^{p_i(x)}\, \mathrm{d}x + \int_{B_r(x_0)} \lambda(x) \left(\frac{1}{q_2^-} - \frac{1}{q_1(x)}\right) |u_n|^{q_1(x)}\, \mathrm{d}x \\
&\geq \left(\frac{1}{P_+^+} - \frac{1}{q_2^-}\right) \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\, \mathrm{d}x - \lambda^\star \left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right) \int_{B_r(x_0)} |u_n|^{q_1(x)}\, \mathrm{d}x.
\end{aligned}
$$

For each $i \in \{1, \ldots, N\}$ and n we define

$$\alpha_{i,n} = \begin{cases} P_+^+ & \text{if } \left|\partial_{x_i} u_n\right|_{p_i(\cdot)} < 1 \\ P_-^- & \text{if } \left|\partial_{x_i} u_n\right|_{p_i(\cdot)} > 1. \end{cases}$$

Using relations (3), (4) and (15) we infer that for n large enough we have

$$\begin{aligned} 1 + \overline{c} + \|u_n\|_{\vec{p}(\cdot)} &\geq \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} u_n\right|^{p_i(x)} \mathrm{d}x - \lambda^\star \left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right) \int_{B_r(x_0)} |u_n|^{q_1(x)} \mathrm{d}x \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{i=1}^{N} \left|\partial_{x_i} u_n\right|_{p_i(\cdot)}^{\alpha_{i,n}} - D_1 \lambda^\star \left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right) \left(\|u_n\|_{\vec{p}(\cdot)}^{q_1^-} + \|u_n\|_{\vec{p}(\cdot)}^{q_1^+} \right) \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{i=1}^{N} \left|\partial_{x_i} u_n\right|_{p_i(\cdot)}^{P_-^-} \\ &\quad - \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{\{i;\ \alpha_{i,n} = P_+^+\}} \left(\left|\partial_{x_i} u_n\right|_{p_i(\cdot)}^{P_-^-} - \left|\partial_{x_i} u_n\right|_{p_i(\cdot)}^{P_+^+} \right) \\ &\quad - D_3 \left(\|u_n\|_{\vec{p}(\cdot)}^{q_1^-} + \|u_n\|_{\vec{p}(\cdot)}^{q_1^+} \right) \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \frac{1}{N^{P_-^-}} \|u_n\|_{\vec{p}(\cdot)}^{P_-^-} - N \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) - D_3 \left(\|u_n\|_{\vec{p}(\cdot)}^{q_1^-} + \|u_n\|_{\vec{p}(\cdot)}^{q_1^+} \right), \end{aligned} \tag{20}$$

where $D_3 = D_1 \lambda^\star \left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right)$ is a positive constant. Dividing the above inequality by $\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}$ taking into account that relation (14) holds true and passing to the limit as $n \to \infty$ we obtain a contradiction. It follows that (u_n) is bounded in E. This information combined with the fact that E is reflexive implies that there exist a subsequence, still denoted by (u_n), and $u_1 \in E$ such that (u_n) converges weakly to u_1 in E. Since, by Theorem 1 in [20] and (14), the space E is compactly embedded in $L^{q(\cdot)}(\Omega)$, it follows that (u_n) converges strongly to u_1 in $L^{q(\cdot)}(\Omega)$. Then by inequality (2) we deduce

$$\lim_{n\to\infty} \int_\Omega \lambda(x) |u_n|^{q(x)-2} u_n (u_n - u_1) \, \mathrm{d}x = 0.$$

This fact and relation (19) yield

$$\lim_{n\to\infty} \langle I'(u_n), u_n - u_1 \rangle = 0.$$

Thus,

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n) \left(\partial_{x_i} u_n - \partial_{x_i} u_1 \right) \mathrm{d}x = 0. \tag{21}$$

Since (u_n) converges weakly to u_1 in E by relation (21) and by following Lemma 1 we find that actually (u_n) converges strongly to u_1 in E. Then, by relation (19) we have

$$I(u_1) = \overline{c} > 0 \quad \text{and} \quad I'(u_1) = 0,$$

that is, u_1 is a non-trivial weak solution of Equation (7).

Next, let $\lambda^\star > 0$ be defined as in (18) and assume $|\lambda|_{L^\infty(\Omega)} < \lambda^\star$. By Lemma 2(i) it follows that on the boundary of the ball centred at the origin and of radius ρ in E, denoted by $V_\rho(0) = \{w \in E;\ \|w\|_{\vec{p}(\cdot)} < \rho\}$, we have

$$\inf_{\partial V_\rho(0)} I > 0. \tag{22}$$

On the other hand, by Lemma 2(iii), there exists $\varphi \in E$ such that $I(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, with the same notations as in Lemma 2(i) relations (15)–(17) imply that for any $u \in V_\rho(0)$ we have

$$\begin{aligned} I(u) \geq & \left[\alpha_1 \|u\|_{\vec{p}(\cdot)}^{P_+^+} - \alpha_2 |\lambda|_{L^\infty(\Omega)} \left(\|u\|_{\vec{p}(\cdot)}^{q_1^-} + \|u\|_{\vec{p}(\cdot)}^{q_1^+} \right) \right] \\ & + \left[\alpha_1 \|u\|_{\vec{p}(\cdot)}^{P_+^+} - \alpha_2 |\lambda|_{L^\infty(\Omega)} \left(\|u\|_{\vec{p}(\cdot)}^{q_2^-} + \|u\|_{\vec{p}(\cdot)}^{q_2^+} \right) \right], \end{aligned}$$

i.e. I is bounded from below on $V_\rho(0)$.

It follows that

$$-\infty < \underline{c} := \inf_{\overline{V_\rho(0)}} I < 0.$$

We let now $0 < \epsilon < \inf_{\partial V_\rho(0)} I - \inf_{V_\rho(0)} I$. Applying Ekeland's variational principle [30] to the functional $I : \overline{V_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{V_\rho(0)}$ such that

$$\begin{aligned} I(u_\epsilon) &< \inf_{\overline{V_\rho(0)}} I + \epsilon \\ I(u_\epsilon) &< I(u) + \epsilon \|u - u_\epsilon\|_{\vec{p}(\cdot)}, \quad u \neq u_\epsilon. \end{aligned}$$

Since

$$I(u_\epsilon) \leq \inf_{\overline{V_\rho(0)}} I + \epsilon \leq \inf_{V_\rho(0)} I + \epsilon < \inf_{\partial V_\rho(0)} I,$$

we deduce that $u_\epsilon \in V_\rho(0)$. Now, we define $T : \overline{V_\rho(0)} \to \mathbb{R}$ by $T(u) = I(u) + \epsilon\|u - u_\epsilon\|_{\vec{p}(\cdot)}$. It is clear that u_ϵ is a minimum point of T and thus

$$\frac{T(u_\epsilon + t\,v) - T(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in V_1(0)$. The above relation yields

$$\frac{I(u_\epsilon + t\,v) - I(u_\epsilon)}{t} + \epsilon\|v\|_{\vec{p}(\cdot)} \geq 0.$$

Letting $t \to 0$ it follows that $\langle I'(u_\epsilon), v\rangle + \epsilon\|v\|_{\vec{p}(\cdot)} > 0$ and we infer that $\|I'(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $(v_n) \subset V_\rho(0)$ such that

$$I(v_n) \to \underline{c} \quad \text{and} \quad I'(v_n) \to 0. \tag{23}$$

It is clear that (v_n) is bounded in E. Thus, there exists $u_2 \in E$ such that, up to a subsequence, (v_n) converges weakly to u_2 in E. Actually, with similar arguments as those used in the proof that the sequence (u_n) converges strongly to u_1 in E we can show that (v_n) converges strongly to u_2 in E. Thus, by (23),

$$I(u_2) = \underline{c} < 0 \quad \text{and} \quad I'(u_2) = 0, \tag{24}$$

i.e. u_2 is a non-trivial weak solution for problem (7).

Finally, since

$$I(u_1) = \overline{c} > 0 > \underline{c} = I(u_2)$$

we see that $u_1 \neq u_2$. In conclusion, problem (7) has two non-trivial weak solutions. The proof of Theorem 1 is complete. ■

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Appendix

We will prove the following auxiliary result.

PROPOSITION 1 *The norms* $\|\cdot\|_{1,p(\cdot)}$ *and* $\|\cdot\|_{p(\cdot)}$ *are equivalent, provided that* $p(x) \geq 2$ *for any* $x \in \overline{\Omega}$.

Proof In order to show that the two norms given in the hypotheses are equivalent we introduce a third norm, namely,

$$\|u\|_{2,p(\cdot)} = \max_{i\in\{1,\ldots,N\}}\{|\partial_{x_i}u|_{p(\cdot)}\}.$$

We will show that actually all these norms are equivalent.

First, we have

$$\|u\|_{2,p(\cdot)} \leq \|u\|_{p(\cdot)} \leq N\|u\|_{2,p(\cdot)}, \quad \forall\, u \in E.$$

Thus, the norms $\|\cdot\|_{2,p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$ are equivalent.

Next, we will prove that there exists a positive constant C_1 such that

$$\|u\|_{1,p(\cdot)} \leq C_1\|u\|_{2,p(\cdot)}, \quad \forall\, u \in E.$$

In order to verify that, we remember that for any $a_1, \ldots, a_N$ positive reals and any real $s \geq 2$ it holds

$$\left(\frac{1}{N}\sum_{i=1}^{N} a_i^2\right)^{1/2} \leq \left(\frac{1}{N}\sum_{i=1}^{N} a_i^s\right)^{1/s}. \tag{25}$$

The above inequality helps us to obtain the following estimates:

$$\begin{aligned}
\int_\Omega \left[\frac{|\nabla u(x)|}{\|u\|_{2,p(\cdot)}}\right]^{p(x)} \mathrm{d}x &= \int_\Omega \frac{(\sum_{i=1}^N |\partial_{x_i} u(x)|^2)^{p(x)/2}}{\|u\|_{2,p(\cdot)}^{p(x)}} \mathrm{d}x \\
&\leq \int_\Omega \frac{\left(\frac{1}{N}\right)^{1-p(x)/2} \sum_{i=1}^N |\partial_{x_i} u(x)|^{p(x)}}{\|u\|_{2,p(\cdot)}^{p(x)}} \mathrm{d}x \\
&\leq C_2 \sum_{i=1}^N \int_\Omega \left(\frac{|\partial_{x_i} u(x)|}{\|u\|_{2,p(\cdot)}}\right)^{p(x)} \mathrm{d}x \\
&\leq C_2 \sum_{i=1}^N \int_\Omega \left(\frac{|\partial_{x_i} u(x)|}{|\partial_{x_i} u|_{p(\cdot)}}\right)^{p(x)} \mathrm{d}x \\
&\leq C_2 N,
\end{aligned}$$

where C_2 is a positive constant. Thus, we actually found that

$$\int_\Omega \left[\frac{|\nabla u(x)|}{(C_2 N)^{1/p^-} \|u\|_{2,p(\cdot)}}\right]^{p(x)} \mathrm{d}x \leq 1.$$

By this inequality and the definition of the Luxemburg's norm we deduce that

$$\|u\|_{1,p(\cdot)} \leq C_1 \|u\|_{2,p(\cdot)}, \quad \forall\, u \in E,$$

with $C_1 = (C_2 N)^{1/p^-}$.

Finally, we will prove that there exists a positive constant C_3 such that

$$\|u\|_{p(\cdot)} \leq C_3 \|u\|_{1,p(\cdot)}, \quad \forall\, u \in E.$$

Indeed, using again inequality (25) and the definition of the Luxemburg norm we obtain the following estimates

$$\int_\Omega \sum_{i=1}^N \left[\frac{|\partial_{x_i} u(x)|}{\|u\|_{1,p(\cdot)}}\right]^{p(x)} \mathrm{d}x \leq \sum_{i=1}^N \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)}} \mathrm{d}x = N \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)}} \mathrm{d}x \leq N.$$

The above inequality shows that for each $i \in \{1, \ldots, N\}$ we have

$$\frac{1}{N}\int_\Omega \left[\frac{|\partial_{x_i} u(x)|}{\|u\|_{1,p(\cdot)}}\right]^{p(x)} \mathrm{d}x \leq 1,$$

and thus,

$$\int_\Omega \left[\frac{|\partial_{x_i} u(x)|}{N^{1/p^-} \|u\|_{1,p(\cdot)}}\right]^{p(x)} \mathrm{d}x \leq 1.$$

By the above inequality and the definition of the Luxemburg norm we deduce that

$$|\partial_{x_i} u|_{p(\cdot)} \le N^{1/p^-} \|u\|_{1,p(\cdot)}, \quad \forall\, i \in \{1, \ldots, N\}.$$

Summing from $i=1$ to N we get

$$\|u\|_{p(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p(\cdot)} \le C_3 \|u\|_{1,p(\cdot)}, \quad \forall\, u \in E,$$

where $C_3 = N^{(1+p^-)/p^-}$.

We conclude that the norms $\|\cdot\|_{1,p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$ are equivalent. The proof of Proposition 1 is complete. ∎