TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 3, pp. 1095-1110, June 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

MULTIPLICITY RESULTS FOR DOUBLE EIGENVALUE PROBLEMS INVOLVING THE p-LAPLACIAN

Hannelore Lisei[1], Gheorghe Moroşanu and Csaba Varga[2]

Abstract. The existence of multiple nontrivial solutions for two types of double eigenvalue problems involving the p-Laplacian is derived. To prove the existence of at least two nontrivial solutions we use a Ricceri-type three critical point result for non-smooth functions of S. Marano and D. Motreanu [12]. The existence of at least three nontrivial solutions is shown by combining a result of B. Ricceri [17] and a Pucci-Serrin mountain pass type theorem of S. Marano and D. Motreanu [12].

1. INTRODUCTION

Let $h_p : \mathbb{R}^N \to \mathbb{R}^N$ be the homeomorphism defined by $h_p(x) = |x|^{p-2}x$ for all $x \in \mathbb{R}^N$, where $p > 1$ is fixed and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

For $T > 0$, let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a mapping satisfying:

(F_1) for each $M > 0$ there exists some $\alpha_M \in L^1(0,T)$ such that, for a.e. $t \in [0,T]$ and all $x, y \in B_M = \{\xi \in \mathbb{R}^N : |\xi| \le M\}$, it holds

$$|F(t,x) - F(t,y)| \le \alpha_M(t)|x-y|;$$

(F_2) the mapping $F(\cdot, x) : [0,T] \to \mathbb{R}$ is measurable for each $x \in \mathbb{R}^N$ and $F(\cdot, 0) \in L^1(0,T)$;

(F_3) $\displaystyle\lim_{|x|\to\infty} \frac{F(t,x) - F(t,0)}{|x|^p} \le 0$ uniformly for a.e. $t \in [0,T]$.

Let $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty, +\infty]$ be a function having the following properties:

Received July 22, 2007, accepted November 21, 2007.
Communicated by B. Ricceri.
2000 *Mathematics Subject Classification*: 34A60, 35J60, 35J65.
Key words and phrases: Vector p-Laplacian, Critical points, Palais-Smale condition, Multiple solutions.
[1]Research for this article was supported in part by Central European University (CEU) Special Extension Programs. The opinions expressed herein are the author's own and do not necessarily express the views of CEU. Financial support for this work was also given in part by MEdC-ANCS, Grant CEEX M2 Nr. 2983/11.10.2005.
[2]Financial support for this work was given in part by MEdC-ANCS Grant CEEX M3, C3, Nr. 130(12441)/2006.

(J_1) $D(j) = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : j(x, y) < +\infty\} \neq \emptyset$ is a closed convex cone with $D(j) \neq \{(0, 0)\}$;

(J_2) j is a convex and lower semicontinuous (shortly, l.s.c.) function.

Let $\gamma > 0$ be arbitrary. For $\lambda, \mu > 0$ we consider the following double eigenvalue problem involving the p-Laplacian operator:

$$(P_{\lambda,\mu}) \qquad \begin{cases} -[h_p(u')]' + \gamma h_p(u) \in \lambda \bar{\partial} F(t, u) \text{ a.e. } t \in [0, T], \\ \Big(h_p(u')(0), -h_p(u')(T)\Big) \in \mu \partial j(u(0), u(T)), \end{cases}$$

where $u : [0, T] \to \mathbb{R}^N$ is of class C^1 and $h_p(u')$ is absolutely continuous. Note, that $\bar{\partial} F(t, \eta)$ denotes the generalized gradient (in the sense of Clarke) of $F(t, \cdot)$ at $\eta \in \mathbb{R}^N$, while ∂j denotes the subdifferential of j in the sense of convex analysis.

Our approach to problem $(P_{\lambda,\mu})$ is a variational one and it relies on results concerning Motreanu-Panagiotopoulos type functionals (see for example in [13] and [14]), which are extensions of the critical point theory of Szulkin type functionals [18].

Previous results concerning p-Laplacian systems with various types of boundary conditions have been obtained by R. Manásevich and J. Mawhin [8], [9], J. Mawhin [10], [11], L. Gasinski and N. Papageorgiu [4], P. Jebelean and G. Moroşanu [6], [7]. As far as we know, eigenvalue problems for differential inclusions involving the p-Laplacian and having mixed boundary conditions where not studied yet. Eigenvalue problems with no boundary conditions were investigated in the books [13],[14] (see also the references therein).

In order to obtain the *existence of multiple solutions* for problem $(P_{\lambda,\mu})$ we impose some further assumptions on F:

(F_4) $\lim\limits_{|x| \to 0} \dfrac{F(t, x) - F(t, 0)}{|x|^p} \leq 0$ uniformly for a.e. $t \in [0, T]$;

(F_5) there exists $s_0 \in \mathbb{R}^N$ such that $\displaystyle\int_0^T \Big(F(t, s_0) - F(t, 0)\Big) dt > 0$.

P. Jebelean and G. Moroşanu [6] proved the existence of a nontrivial solution for a differential inclusion problem of the type $(P_{\lambda,\mu})$ by using "mountain pass theorems". Our paper completes their results by proving the *existence of at least two nontrivial solutions* for a first type of double eigenvalue problem and the *existence of at least three nontrivial solutions* for a second type of double eigenvalue problem. For this, we need assumptions on the behavior around zero and close to infinity of the function F (see (F_3), (F_4), (F_5)). The two types of problems $(P_{\lambda,\mu})$ rely on different assumptions for the function j, and for this reason we use different tools for their investigation.

The main tool for the first type problem is a Ricceri-type three critical point result for non-smooth functions of S. Marano and D. Motreanu [12, Theorem 3.1]. For the second type problem we use a recent result of B. Ricceri [17, Theorem 4] concerning the existence of multiple solutions and a Pucci-Serrin mountain pass type theorem of S. Marano and D. Motreanu [12, Corollary 2.1].

This paper is organized as follows: in Section 2, there are introduced some notations and important preliminary results for problem $(P_{\lambda,\mu})$. Then, in Section 3 it is proved the existence of at least two nontrivial solutions for the first type double eigenvalue problem $(P_{\lambda,\mu})$ and in Section 4 we complete the results of Section 3 by showing the existence of at least three nontrivial solutions for the second type double eigenvalue problem $(P_{\lambda,\mu})$. Finally, Section 5 contains important results from variational calculus concerning the critical point theory, which are used in our investigations.

2. Notations and Preliminary Results

Let $W^{1,p} = W^{1,p}(0,T;\mathbb{R}^N)$ be the usual Sobolev space equipped with the norm

$$\|u\|_\eta = \Big(\|u'\|^p_{L^p} + \eta\|u\|^p_{L^p}\Big)^{1/p},$$

where $\eta > 0$, and $\|\cdot\|_{L^p}$ is the norm of $L^p = L^p(0,T;\mathbb{R}^N)$

$$\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{1/p}.$$

We consider $C = C([0,T];\mathbb{R}^N)$ endowed with the norm

$$\|u\|_C = \max\{|u(t)| : t \in [0,T]\}.$$

For $\gamma > 0$, we consider $\varphi_\gamma : W^{1,p} \to \mathbb{R}$ defined by

$$\varphi_\gamma(u) := \frac{1}{p}\Big(\|u'\|^p_{L^p} + \gamma\|u\|^p_{L^p}\Big) \text{ for all } u \in W^{1,p}.$$

Note, that φ_γ is convex and $\varphi_\gamma \in C^1(W^{1,p};\mathbb{R})$ with

$$\langle \varphi'_\gamma(u), v\rangle = \int_0^T (h_p(u'), v')dt + \gamma \int_0^T (h_p(u), v)dt \text{ for all } u, v \in W^{1,p}.$$

We define the function $J : W^{1,p} \to]-\infty, +\infty]$ by

$$J(u) = j(u(0), u(T)) \text{ for all } u \in W^{1,p}.$$

J is a proper, convex and l.s.c. function. Note, that

$$D(J) = \{u \in W^{1,p} : (u(0), u(T)) \in D(j)\}.$$

We introduce the constant $\gamma_1 = \gamma_1(p, \gamma) > 0$ by setting

$$\gamma_1 = \inf \left\{ \frac{\|u'\|_{L^p}^p + \gamma \|u\|_{L^p}^p}{\|u\|_{L^p}^p} : u \in W^{1,p} \setminus \{0\}, u \in D(J) \right\}.$$

By computation one has

$$2^{-1/p} \|u\|_{\gamma_1} \leq (\|u'\|_{L^p}^p + \gamma \|u\|_{L^p}^p)^{1/p} \leq \|u\|_{\gamma_1} \text{ for all } u \in D(J). \tag{2.1}$$

We consider the functional $\hat{\mathcal{F}} : C \to \mathbb{R}$ defined by

$$\hat{\mathcal{F}}(v) = -\int_0^T F(t, v)dt + \int_0^T F(t, 0)dt \text{ for all } v \in C$$

and $\mathcal{F} : W^{1,p} \to \mathbb{R}$ defined by $\mathcal{F} = \hat{\mathcal{F}}\Big|_{W^{1,p}}$. The functional $\mathcal{F}$ is sequentially weakly continuous, since the embedding $W^{1,p} \hookrightarrow C$ is compact.

Note that for $1 \leq r < p$ and $p < q < p^*$ the embeddings $L^p \hookrightarrow L^r$, $W^{1,p} \hookrightarrow L^q$, $W^{1,p} \hookrightarrow C$ are continuous, hence there exist constants $C_{r,p}, \hat{C}_{q,p}, \hat{c} > 0$ such that

$$\|u\|_{L^r} \leq C_{r,p} \|u\|_{L^p}, \quad \|u\|_{L^q} \leq \hat{C}_{q,p} \|u\|_{W^{1,p}}, \ \|u\|_C \leq \hat{c} \|u\|_{W^{1,p}} \text{ for all } u \in W^{1,p}.$$

Let $\mathcal{E} : [0, \infty) \times [0, \infty) \times W^{1,p} \to]-\infty, \infty]$ be defined by

$$\mathcal{E}(\lambda, \mu, u) = \varphi_\gamma(u) + \lambda \mathcal{F}(u) + \mu J(u).$$

The functional $\mathcal{E}$ is of Motreanu-Panagiotopoulos type.

Proposition 2.1. [6, Proposition 3.2]. *Assume that* $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* (F_1) *and* (F_2) *and* $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty, +\infty]$ *satisfies* (J_1) *and* (J_2)*. If* $u \in W^{1,p}$ *is a critical point of* $\mathcal{E}(\lambda, \mu, \cdot)$ *(in the sense of Definition 5.1), then* u *is a solution of* $(P_{\lambda,\mu})$*.*

Remark 2.1. Let $\varepsilon > 0$ be arbitrary. From (F_1), (F_2) and (F_3) it follows that there exists $\delta_1 > 0$ (depending on ε) such that

$$F(t, x) - F(t, 0) \leq \varepsilon |x|^p + \alpha_{\delta_1}(t)\delta_1 \quad \text{ for all } x \in \mathbb{R}^N, \text{ a.e. } t \in [0, T].$$

Then

$$\mathcal{F}(u) \geq -\varepsilon \|u\|_{L^p}^p - \delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} \quad \text{ for all } u \in W^{1,p}. \tag{2.2}$$

Proposition 2.2. *Assume that* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* (F_1), (F_2) *and* (F_3) *and that* $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty, +\infty]$ *satisfies* (J_1) *and* (J_2). *Then the following properties hold:*

(1) $\mathcal{E}(\lambda, \mu, \cdot)$ *is weakly sequentially lower semicontinuous on* $W^{1,p}$ *for each* $\lambda > 0$, $\mu \geq 0$;

(2) $\lim\limits_{\|u\|_{\gamma_1} \to +\infty} \mathcal{E}(\lambda, \mu, u) = +\infty$ *for each* $\lambda > 0, \mu \geq 0$;

(3) $\mathcal{E}(\lambda, \mu, \cdot)$ *satisfies the* (PS) *condition for each* $\lambda, \mu > 0$.

Proof. (1) The function $\mathcal{E}(\lambda, \mu, \cdot)$ is weakly sequentially l.s.c on $W^{1,p}$, because $\mathcal{F}$ is weakly sequentially l.s.c., while φ_γ and J are convex and l.s.c., hence they are also weakly sequentially l.s.c.

(2) First observe that

$$\|u\|_{L^p}^p \leq \frac{1}{\gamma_1} \|u\|_{\gamma_1}^p \text{ for all } u \in W^{1,p}.$$

In (2.2) we choose $\varepsilon < \frac{\gamma_1}{2\lambda p}$. Using that the embedding $L^p \hookrightarrow L^1$ is continuous and that (2.1) holds, we have for all $u \in D(J)$

$$\begin{aligned} \mathcal{E}(\lambda, \mu, u) \geq\ & \frac{1}{p}\Big(\|u'\|_{L^p}^p + \gamma \|u\|_{L^p}^p\Big) - \lambda\varepsilon \|u\|_{L^p}^p - \lambda\delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} + \mu J(u) \\ \geq\ & \frac{\gamma_1 - 2\varepsilon\lambda p}{2\gamma_1 p} \|u\|_{\gamma_1}^p - \lambda\delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} + \mu J(u). \end{aligned}$$

Since J is convex and l.s.c. it is bounded from below by an affine functional and then there exist constants $c_1, c_2, c_3 > 0$ such that for all $u \in D(J)$

$$\mathcal{E}(\lambda, \mu, u) \geq \frac{\gamma_1 - 2\varepsilon\lambda p}{2\gamma_1 p} \|u\|_{\gamma_1}^p - \lambda\delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} - c_1|u(0)| - c_2|u(T)| - c_3.$$

By the continuity of the embedding $W^{1,p} \hookrightarrow C$ we have for all $u \in W^{1,p}$

$$\mathcal{E}(\lambda, \mu, u) \geq c_4 \|u\|_{\gamma_1}^p - c_5 \|u\|_{\gamma_1} - c_6,$$

where $c_4, c_5, c_6 > 0$ are constants. Since, $1 < p$ it follows that $\mathcal{E}(\lambda, \mu, \cdot) \to +\infty$ when $\|u\|_{\gamma_1} \to +\infty$.

(3) Let $\{u_n\}$ in $W^{1,p}$ be a sequence satisfying $\mathcal{E}(\lambda, \mu, u_n) \to c$ and

$$\lambda\mathcal{F}^0(u_n; v - u_n) + \varphi_\gamma(v) - \varphi_\gamma(u_n) + \mu J(v) - \mu J(u_n) \geq -\varepsilon_n \|v - u_n\|_{\gamma_1}, \forall v \in W^{1,p},$$

where $\{\varepsilon_n\} \subset [0, \infty)$ with $\varepsilon_n \to 0$. We have a subsequence $\{u_n\} \subset D(J)$ (we just eliminate the finite number of elements of the sequence which do not belong to $D(J)$), since $\mu > 0$ and $\mathcal{E}(\lambda, \mu, u_n) \to c$.

But $\mathcal{E}(\lambda, \mu, \cdot)$ is coercive, this implies that $\{u_n\}$ is bounded in $W^{1,p}$. The embedding $W^{1,p} \hookrightarrow C$ is compact, then we can find a subsequence, which we still denote by $\{u_n\}$, which is weakly convergent to a point $u \in W^{1,p}$ and strongly in C.

In the above inequality we take $v = u_n + s(u - u_n)$, with $s > 0$, then divide both sides of the inequality by s and let $s \searrow 0$, to obtain

$$\lambda \mathcal{F}^0(u_n; u-u_n) + \varphi'_\gamma(u_n; u-u_n) + \mu J'(u_n; u-u_n) \geq -\varepsilon_n \|u-u_n\|_{\gamma_1}, \qquad \forall n \in \mathbb{N}.$$

By the upper semicontinuity of $\hat{\mathcal{F}}^0$ (see [14], Chapter 1), it follows that

$$\liminf_{n\to\infty} \Big(\varphi'_\gamma(u_n; u-u_n) + \mu J'(u_n; u-u_n) \Big) \geq 0.$$

By Lemma 4.1 in [6] it follows that $\{u_n\}$ converges strongly to $u \in W^{1,p}$. ■

Remark 2.2. From (F_1), (F_2), (F_3) and (F_4) it follows that for each $\varepsilon > 0$ there exist $\delta_\varepsilon, \bar{\delta}_\varepsilon > 0$ such that

$$F(t,x) - F(t,0) \leq \varepsilon |x|^p + \frac{\alpha_{\delta_\varepsilon}(t)}{\bar{\delta}_\varepsilon^{r-1}} |x|^r \qquad \text{for all } x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T],$$

where $r \geq 1$. Then, by using the continuity of the embedding $W^{1,p} \hookrightarrow C$ we get

$$\mathcal{F}(u) \geq -\varepsilon \|u\|_{L^p}^p - \frac{\hat{c}^r \|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\bar{\delta}_\varepsilon^{r-1}} \|u\|_\gamma^r \qquad \text{for all } u \in W^{1,p}. \tag{2.3}$$

Remark 2.3. If $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies (F_1) and (F_4), then $0 \in \bar{\partial} F(t,0)$ for a.e. $t \in [0,T]$. In order to prove this property, let $x \in \mathbb{R}^N$ be fixed. From (F_4) it follows that there exists $\delta > 0$ such that

$$F(t,z) - F(t,0) \leq |z|^p \text{ for each } |z| < \delta \text{ and a.e. } t \in [0,T]. \tag{2.4}$$

But

$$(-F)^0(t,0;x) = \lim_{\varepsilon \searrow 0} \sup_{\substack{0<|w|<\varepsilon \\ 0<s<\varepsilon}} \frac{-F(t,w+sx) + F(t,w)}{s}.$$

Let $\varepsilon > 0$ be fixed and let $\{w_n\}$ be a sequence in $\mathbb{R}^N$ such that $|w_n| \searrow 0$ and $|w_n| < \varepsilon$ for all $n \in \mathbb{N}$. Then for $0 < s < \varepsilon$ and $n \in \mathbb{N}$ we have

$$\frac{-F(t,w_n+sx) + F(t,w_n)}{s} \leq \sup_{\substack{0<|w|<\varepsilon \\ 0<s<\varepsilon}} \frac{-F(t,w+sx) + F(t,w)}{s}.$$

Since $F(t,\cdot)$ is continuous (see (F_1)), we get for $n \to \infty$

$$\frac{-F(t,sx)+F(t,0)}{s} \leq \sup_{\substack{0<|w|<\varepsilon \\ 0<s<\varepsilon}} \frac{-F(t,w+sx)+F(t,w)}{s},$$

when $0 < s < \varepsilon$. By (2.4) it follows that

$$-s^{p-1}|x|^p \leq \sup_{\substack{0<|w|<\varepsilon \\ 0<s<\varepsilon}} \frac{-F(t,w+sx)+F(t,w)}{s},$$

when s is small enough such that $|sx| < \delta$. Finally we take $\varepsilon \searrow 0$ and get

$$0 \leq (-F)^0(t,0;x) = F^0(t,0;-x) \text{ for all } x \in \mathbb{R}^N.$$

This implies, $0 \in \bar{\partial} F(t,0)$ for a.e. $t \in [0,T]$.

3. First Type Problem

In order to obtain the existence of at least two nontrivial solutions for $(P_{\lambda,\mu})$ we impose some further assumptions on the convex function $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty,+\infty]$ which satisfies (J_1) and (J_2):

(J_3) $j(0,0)=0$, $j(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Theorem 3.1. *Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying $(F_1)-(F_5)$ and let $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty,+\infty]$ be a function satisfying $(J_1)-(J_3)$. Then for each fixed $\mu > 0$, there exists an open interval $\Lambda_\mu \subset]0,+\infty[$ such that for each $\lambda \in \Lambda_\mu$, the problem $(P_{\lambda,\mu})$ has at least two nontrivial solutions.*

Proof. Let $\mu > 0$ be fixed. We define the function $g :]0,+\infty[\to \mathbb{R}$, by

$$g(t) = \sup\{-\mathcal{F}(u) \ : \ \varphi_\gamma(u) + \mu J(u) \leq t\}, \quad \text{for all} \quad t > 0.$$

Using (2.3) for $r \in]p,p^*[$ it follows that for all $u \in W^{1,p}$ we have

$$-\mathcal{F}(u) \leq \frac{\varepsilon}{\gamma}\|u\|_\gamma^p + \frac{\hat{c}^r \|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\bar{\delta}_\varepsilon^{r-1}} \|u\|_\gamma^r.$$

Since $p < r$, this implies

$$\lim_{t\to 0^+} \frac{g(t)}{t} = 0.$$

Using (F_5) we define $u_0(t) = s_0$ for a.e. $t \in [0,T]$. Then, $u_0 \in W^{1,p} \setminus \{0\}$ and $-\mathcal{F}(u_0) > 0$. Due to the convergence relation above, it is possible to choose a real number t_0 such that $0 < t_0 < \varphi_\gamma(u_0) + \mu J(u_0)$ and

$$\frac{g(t_0)}{t_0} < [\varphi_\gamma(u_0) + \mu J(u_0)]^{-1} \cdot (-\mathcal{F}(u_0)).$$

We choose $\rho_0 > 0$ such that

$$g(t_0) < \rho_0 < [\varphi_\gamma(u_0) + \mu J(u_0)]^{-1} \cdot (-\mathcal{F}(u_0))t_0. \tag{3.1}$$

We apply Theorem 5.2 to the space $W^{1,p}$, the interval $\Lambda =]0, +\infty[$ and the functions $\mathcal{G}, \mathcal{H} : W^{1,p} \to \mathbb{R}, h : \Lambda \to \mathbb{R}$ defined by

$$\mathcal{G}(u) = \varphi_\gamma(u), \psi(u) = \mu J(u), \mathcal{H}(u) = \mathcal{F}(u), h(\lambda) = \rho_0\lambda.$$

By Proposition 2.2 the assumption (a) from Theorem 5.2 is fulfilled.

We prove now the minimax inequality

$$\begin{aligned}&\sup_{\lambda\in\Lambda} \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big)\\ &< \inf_{u\in W^{1,p}} \sup_{\lambda\in\Lambda} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big).\end{aligned}$$

The function

$$\lambda \mapsto \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big)$$

is upper semicontinuous on Λ. Since

$$\inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big) \le \varphi_\gamma(u_0) + \mu J(u_0) + \lambda\mathcal{F}(u_0) + \rho_0\lambda$$

and $\rho_0 < -\mathcal{F}(u_0)$, it follows that

$$\lim_{\lambda\to+\infty} \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big) = -\infty.$$

Thus we can find $\overline{\lambda} \in \Lambda$ such that

$$\begin{aligned}\beta_1 :&= \sup_{\lambda\in\Lambda} \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big)\\ &= \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \overline{\lambda}\mathcal{F}(u) + \rho_0\overline{\lambda}\Big).\end{aligned}$$

In order to prove that $\beta_1 < t_0$, we distinguish two cases:

I. If $0 \le \overline{\lambda} < \frac{t_0}{\rho_0}$, we have

$$\beta_1 \le \varphi_\gamma(0) + \mu J(0) + \overline{\lambda}\mathcal{F}(0) + \rho_0\overline{\lambda} = \overline{\lambda}\rho_0 < t_0.$$

II. If $\overline{\lambda} \ge \frac{t_0}{\rho_0}$, then we use $\rho_0 < -\mathcal{F}(u_0)$ and the inequality (3.1) to get

$$\eta_1 \le \varphi_\gamma(u_0)+\mu J(u_0)+\overline{\lambda}\mathcal{F}(u_0)+\rho_0\overline{\lambda} \le \varphi_\gamma(u_0)+\mu J(u_0)+\frac{t_0}{\rho_0}(\rho_0+\mathcal{F}(u_0)) < t_0.$$

From $g(t_0) < \rho_0$ it follows that for all $u \in W^{1,p}$ with $\varphi_\gamma(u) + \mu J(u) \le t_0$ we have $-\mathcal{F}(u) < \rho_0$. Hence

$$t_0 \le \inf\{\varphi_\gamma(u) + \mu J(u) : \ -\mathcal{F}(u) \ge \rho_0\}.$$

On the other hand,

$$\begin{aligned}\beta_2 &= \inf_{u \in W^{1,p}} \sup_{\lambda \in \Lambda}(\varphi_\gamma(u) + \mu J(u) + \lambda \mathcal{F}(u) + \rho_0 \lambda)\\ &= \inf\{\varphi_\gamma(u) + \mu J(u) : -\mathcal{F}(u) \ge \rho_0\}.\end{aligned}$$

We conclude that

$$\beta_1 < t_0 \le \beta_2.$$

Hence, assumption (b) from Theorem 5.2 holds. Then, by Theorem 5.2 it follws that there exists an open interval $\Lambda_\mu \subseteq]0, \infty)$ such that for each $\lambda \in \Lambda_\mu$ the function $\varphi_\gamma + \mu J + \lambda \mathcal{F}$ has at least three critical points in $W^{1,p}$. By Proposition 2.1 it follows that these critical points are solutions of $(P_{\lambda,\mu})$. Since $0 \in \bar{\partial} F(t, 0)$ for a.e. $t \in [0, T]$, we get that at least two of the above solutions are nontrivial. ∎

Remark 3.1. The two conditions from (J_3) can be replaced by

(J_3') $j(x, y) \ge j(0, 0)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Then, all the proofs above can be adapted by considering

$$J(u) = j(u(0), u(T)) - j(0, 0).$$

Corollary 3.1. *Let $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying $(F_1) - (F_5)$ and let $b : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a positive, convex and Gâteaux differentiable function with $b(0, 0) = 0$. Assume that $S \subset \mathbb{R}^N \times \mathbb{R}^N$ is a nonempty closed convex cone with $S \ne \{(0, 0)\}$, whose normal cone we denote by N_S. Then for each fixed $\gamma, \mu > 0$, there exists an open interval $\Lambda_0 \subset]0, +\infty[$ such that for each $\lambda \in \Lambda_0$, the following problem*

$$(\hat{P}_{\lambda,\mu}) \quad \begin{cases} -[h_p(u')]' + \gamma h_p(u) \in \lambda \bar{\partial} F(t, u) \text{ a.e. } t \in [0, T], \\ (u(0), u(T)) \in S, \\ \Big(h_p(u')(0), -h_p(u')(T)\Big) \in \mu \nabla b(u(0), u(T)) + \mu N_S(u(0), u(T)), \end{cases}$$

has at least two nontrivial solutions.

Proof. The statement follows by applying Theorem 3.1 to the function F and the convex function $j : \mathbb{R}^N \times \mathbb{R}^N \to] - \infty, +\infty]$ defined by

$$j(x, y) = b(x, y) + I_S(x, y), \text{ for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where

$$I_S(x,y) = \begin{cases} 0, \text{ if } (x,y) \in S \\ +\infty, \text{ if } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus S, \end{cases}$$

is the indicator function of the cone S.

Note, that in this case $D(j) = S$ and j satisfies the conditions $(J_1) - (J_3)$. Moreover,

$$\partial j(x,y) = \nabla b(x,y) + \partial I_S(x,y) = \nabla b(x,y) + N_S(x,y) \text{ for all } (x,y) \in S. \quad \blacksquare$$

Example 3.1. We give an example of a function F that satisfies the assumptions (F_1) to (F_5): Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$F(t,x) = f(t) - \min\{|x|^{p+\alpha}, |x|^{p-\beta} + 1\} \text{ for all } t \in [0,T], x \in \mathbb{R}^N,$$

where $\alpha > 0, \beta \in]0,p[, f \in L^1(0,T)$.

Various possible choices of b and S from Corollary 3.1 recover some classical boundary conditions. For instance:

(a) $b = 0$ and $S = \{(x,x) : x \in \mathbb{R}^N\}$ we get periodic boundary conditions $u(0) = u(T), u'(0) = u'(T)$;

(b) $b = 0$ and $S = \mathbb{R}^N \times \mathbb{R}^N$ we get Neumann type boundary conditions $u'(0) = u'(T) = 0$;

(c) $b(z) = \frac{1}{2}(Az,z)_{R^{2N}}$, $z \in \mathbb{R}^{2N}$, where A is a symmetric, positive $2N \times 2N$ real valued matrix, and $S = \mathbb{R}^N \times \mathbb{R}^N$; we get the following mixed boundary conditions

$$\begin{pmatrix} h_p(u')(0) \\ -h_p(u')(T) \end{pmatrix} = A \begin{pmatrix} u(0) \\ u(T) \end{pmatrix}.$$

For these choices of F, b and S it follows by Corollary 3.1 that for each fixed $\gamma, \mu > 0$, there exists an open interval $\Lambda_0 \subset]0,+\infty[$ such that for each $\lambda \in \Lambda_0$ the problem $(\hat{P}_{\lambda,\mu})$ has at least two nontrivial solutions.

4. Second Type Problem

Theorem 4.1. *Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying $(F_1) - (F_5)$ and let $j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a convex function. Then, there exist a non-degenerate compact interval $[a,b] \subset]0,+\infty[$ and a number $\sigma_0 > 0$ such that for every $\lambda \in [a,b]$ there exists $\mu_0 > 0$ such that for each $\mu \in]0,\mu_0[$, the problem $(P_{\lambda,\mu})$ has at least three solutions with norms less than σ_0. Moreover, if $0 \notin \partial j(0,0)$, then these solutions are nontrivial.*

Proof. We define the function $g :]0, +\infty[\to \mathbb{R}$, by

$$g(t) = \sup \{-\mathcal{F}(u) \ : \ \varphi_\gamma(u) \le t \ \}, \quad \text{for all} \quad t > 0.$$

Using (2.3) for $r \in]p, p^*[$ it follows that for all $u \in W^{1,p}$ we have

$$-\mathcal{F}(u) \le \frac{\varepsilon}{\gamma}\|u\|_\gamma^p + \frac{\hat{c}^r \|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\bar{\delta}_\varepsilon^{r-1}}\|u\|_\gamma^r.$$

Since $p < r$, this implies

$$\lim_{t \to 0^+} \frac{g(t)}{t} = 0.$$

As in the proof of Theorem 3.1, by (F_5) there exists $u_0 \in W^{1,p} \setminus \{0\}$ such that $-\mathcal{F}(u_0) > 0$. Due to the convergence relation above, it is possible to choose a real number t_0 such that $0 < t_0 < \varphi_\gamma(u_0)$ and

$$\frac{g(t_0)}{t_0} < [\varphi_\gamma(u_0)]^{-1} \cdot (-\mathcal{F}(u_0)).$$

We choose $\rho_0 > 0$ such that

$$g(t_0) < \rho_0 < [\varphi_\gamma(u_0)]^{-1} \cdot (-\mathcal{F}(u_0))t_0.$$

We apply Theorem 5.3 to the space $W^{1,p}$, the interval $I =]0, +\infty[$ and the function $\Psi : W^{1,p} \times I \to \mathbb{R}$ defined by

$$\Psi(u, \lambda) = \varphi_\gamma(u) + \lambda\left(\rho_0 + \mathcal{F}(u)\right), \text{ for all } (u, \lambda) \in W^{1,p} \times I$$

and $\Phi : W^{1,p} \to \mathbb{R}$ by

$$\Phi(u) = J(u) \text{ for all } u \in W^{1,p}.$$

Clearly, by Proposition 2.2 $\Psi(\cdot, \lambda)$ and Φ are sequentially weakly l.s.c. for all $u \in W^{1,p}$. Moreover, $\Psi(\cdot, \lambda)$ is continuous (the norm φ_γ and $\mathcal{F}$ are continuous functions), coercive (by Proposition 2.2), and obviously $\Psi(u, \cdot)$ is concave for all $u \in W^{1,p}$.

By the same technique as in the proof of Theorem 3.1 we prove the minimax inequality

$$\sup_{\lambda \in I} \inf_{u \in W^{1,p}} \Psi(u, \lambda) < \inf_{u \in W^{1,p}} \sup_{\lambda \in I} \Psi(u, \lambda).$$

Note, that the role of the function $\varphi_\gamma + J + \lambda\mathcal{F} + \rho_0\lambda$ from Theorem 3.1 is now replaced by $\Psi(\cdot, \lambda)$.

We can apply Theorem 5.3. Fix $\delta > \eta_1$, and for every $\lambda \in I$ denote

$$S_\lambda = \left\{u \in W^{1,p} \ : \ \Psi(u, \lambda) < \delta\right\}.$$

There exists a non-empty open set $I_0 \subset]0, +\infty[$ with the following property: for every $\lambda \in I_0$ there exists $\lambda_0 > 0$, such that for each $\mu \in]0, \mu_0[$, the functional

$$u \to \Psi(u, \lambda) + \mu\Phi(u)$$

has at least two local minima lying in the set S_λ. Let $[a, b] \subset I_0$ be a non-degenerate compact interval.

We prove now the assertion of our theorem: Let $\lambda \in [a, b]$ be a real number. From what stated above, there exists $\mu_0 > 0$ such that for all $\mu \in]0, \mu_0[$ the functional $\mathcal{E}(\lambda, \mu, \cdot)$ admits at least two local minima $u^1_{\lambda,\mu},\ u^2_{\lambda,\mu} \in S_\lambda$, therefore by Proposition 5.1 (for $\mathcal{G}(u) = \lambda\mathcal{F}(u), \psi(u) = \varphi_\gamma(u) + \mu J(u), u \in W^{1,p}$) these are critical points of $\mathcal{E}(\lambda, \mu, \cdot)$.

Observe that

$$S := \bigcup_{\lambda\in[a,b]} S_\lambda \subseteq S_a \cup S_b.$$

Since $\Psi(\cdot, \lambda)$ is coercive (see Proposition 2.2 applied for $\mathcal{E}(\lambda, 0, \cdot)$), the latter sets are bounded, hence S is bounded as well. By choosing $\sigma_0 > \sup_{u\in S} \|u\|_{\gamma_1}$, we get

$$\|u^1_{\lambda,\mu}\|_{\gamma_1}, \|u^2_{\lambda,\mu}\|_{\gamma_1} < \sigma_0.$$

To prove the existence of a third critical point for $\mathcal{E}(\lambda, \mu, \cdot)$, we apply Proposition 5.2 (for $\mathcal{G}(u) = \lambda\mathcal{F}(u) + \varphi_\gamma(u) + \mu J(u), \psi(u) = 0, u \in W^{1,p}$; note that, since J is convex and continuous, it is then also locally Lipschitz), since the (PS) condition holds by Proposition 2.2. Finally, by Proposition 2.1 it follows that these critical points are solutions of $(P_{\lambda,\mu})$.

Obviously, if $0 \notin \partial j(0, 0)$, then each solution is nontrivial. ■

Example 4.1. We give an example of functions F and j that satisfy the assumptions of Theorem 4.1: Let $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$F(t, x) = -f(t) \cdot \min\{|x|^{p+\alpha}, |x|^{p-\beta} + 1\} \text{ for all } t \in [0, T], x \in \mathbb{R}^N,$$

where $\alpha > 0, \beta \in]0, p[, f \in L^1(0, T; \mathbb{R}_+) \setminus \{0\}$, and let $j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be given by

$$j(x, y) = \max\{|(x, y)-(1, 1)|^a+1, |(x, y)-(1, 1)|^b+1\} \text{ for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $a > b \geq 1$ and $(1, 1) \in \mathbb{R}^N \times \mathbb{R}^N$ denotes the vector with all coordinates 1. By Theorem 4.1 it follows that in this case there exist at least three nontrivial solutions for the eigenvalue problem $(P_{\lambda,\mu})$.

5. Appendix - Basic Notions and Results

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $\mathcal{G} : X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|\mathcal{G}(u_1) - \mathcal{G}(u_2)| \leq L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of $\mathcal{G}$ at the point $u \in X$ in the direction $z \in X$ is

$$\mathcal{G}^0(u; z) = \limsup_{w \to u, s \to 0^+} \frac{\mathcal{G}(w + sz) - \mathcal{G}(w)}{s}.$$

The *generalized gradient* (in the sense of Clarke [1]) of $\mathcal{G}$ at $u \in X$ is defined by

$$\bar{\partial}\mathcal{G}(u) = \{x^* \in X^* : \langle x^*, x\rangle \leq \mathcal{G}^0(u; x),\ \forall x \in X\},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X.

Let $\mathcal{G} : X \to \mathbb{R}$ be a locally Lipschitz function, and let $\psi : X \to]-\infty, +\infty]$ be a convex, proper, l.s.c. function.

Definition 5.1. [14]. An element $u \in X$ is said to be a critical point of $\mathcal{E} = \mathcal{G} + \psi$, if

$$\mathcal{G}^0(u; v - u) + \psi(v) - \psi(u) \geq 0, \forall v \in X.$$

In this case, $\mathcal{E}(u)$ is a critical value of $\mathcal{E}$.

In the case of differentiable functions one gets the notion of critical point introduced by A. Szulkin [18].

Definition 5.2. [14]. The functional $\mathcal{E} = \mathcal{G} + \psi$ is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ $(shortly, (PS)_c)$ if every sequence $\{u_n\}$ in X satisfying $\mathcal{E}(u_n) \to c$ and

$$\mathcal{G}^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\|, \forall v \in X,$$

for a sequence $\{\varepsilon_n\} \subset [0, \infty)$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in \mathbb{R}$, $\mathcal{E}$ is said to satisfy the Palais-Smale condition $(shortly, \text{(PS)})$.

Proposition 5.1. [12, Proposition 2.1]. *Each local minimum of $\mathcal{E} = \mathcal{G} + \psi$ is necessarily a critical point of $\mathcal{E}$.*

Theorem 5.2. [12, Theorem 3.1]. *Assume that X is a separable and reflexive Banach space, Λ is a real interval, $\mathcal{G}, \mathcal{H} : X \to \mathbb{R}$ are locally Lipschitz functions and $\psi : X \to]-\infty, +\infty]$ is a convex, proper, l.s.c. function, such that:*

(a) *for every $\lambda \in \Lambda$ the function $\mathcal{G} + \psi + \lambda\mathcal{H}$ fulfils the (PS) condition, together with*

$$\lim_{\|u\| \to +\infty} \Big(\mathcal{G}(u) + \psi(u) + \lambda\mathcal{H}(u) \Big) = +\infty;$$

(b) *there exists a continuous concave function $h : \Lambda \to \mathbb{R}$ satisfying*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} \Big(\mathcal{G}(u) + \psi(u) + \lambda\mathcal{H}(u) + h(\lambda) \Big)$$
$$< \inf_{u \in X} \sup_{\lambda \in \Lambda} \Big(\mathcal{G}(u) + \psi(u) + \lambda\mathcal{H}(u) + h(\lambda) \Big).$$

Then, there is an open interval $\Lambda_0 \subseteq \Lambda$ such that for each $\lambda \in \Lambda_0$ the function $\mathcal{G} + \psi + \lambda\mathcal{H}$ has at least three critical points in X.

The following result is proved by Marano and Motreanu and it generalizes results of P. Pucci, J. Serrin [16]:

Proposition 5.2. [12, Corollary 2.1]. *Let $I = \mathcal{G} + \psi$ satisfying the Palais-Smale condition (PS). If $\mathcal{E}$ has two local minima $u_0, u_1 \in X$, then it admits at least three critical points.*

The main tool in our investigations is the result of B. Ricceri [17, Theorem 4], which we state for the reader's convenience in a slightly modified form (adapted for the weak topology), suitable for our purposes:

Theorem 5.3. *Let X be a real, reflexive, separable Banach space, let $I \subseteq \mathbb{R}$ be an interval, and let $\Psi : X \times I \to]-\infty, +\infty]$ be a function satisfying the following conditions:*

(1) $\Psi(x, \cdot)$ *is concave in I for all $x \in X$;*

(2) $\Psi(\cdot, \nu)$ *is upper semicontinous, coercive and sequentially weakly lower semicontinuous in X for all $\nu \in I$;*

(3) $\eta_1 := \sup_{\nu \in I} \inf_{x \in X} \Psi(x, \nu) < \inf_{x \in X} \sup_{\nu \in I} \Psi(x, \nu) =: \eta_2$.

Then, for each $\delta > \eta_1$ there exists a non-empty open set $I_0 \subset I$ with the following property: for every $\nu \in I_0$ and every sequentially weakly l.s.c. function $\Phi : X \to \mathbb{R}$, there exists $\tau_0 > 0$ such that, for each $\tau \in]0, \tau_0[$, the function $\Psi(\cdot, \nu) + \tau\Phi(\cdot)$ has at least two local minima lying in the set $\{x \in X \ : \ \Psi(x, \nu) < \delta\}$.

ACKNOWLEDGMENTS

The authors would like to address special thanks to Prof. B. Ricceri for his useful hints.

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Hannelore Lisei and Csaba Varga
Babeş-Bolyai University,
Faculty of Mathematics and Computer Science,
Str. Kogalniceanu nr. 1,
RO-400084 Cluj-Napoca, Romania
E-mail: hanne@math.ubbcluj.ro
csvarga@cs.ubbcluj.ro

Gheorghe Moroşanu
Department of Mathematics and Its Applications,
Central European University,
Nador u. 9, H-1051 Budapest,
Hungary
E-mail: morosanug@ceu.hu