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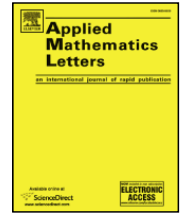
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A necessary and sufficient condition for input identifiability for linear time-invariant systems

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ABSTRACT

A necessary and sufficient condition for input identifiability for linear autonomous systems is given. The result is based on a finite iterative process and its proof relies on elementary arguments involving matrices, finite dimensional linear spaces, Gronwall's lemma, and linear differential systems. Our condition is equivalent to the classical condition involving the geometrical concept of controlled invariant [V. Basile, G. Marro, Controlled and Conditioned Invariants in Linear System Theory, Prentice Hall, Englewood Cliffs, NJ, 1992, p. 237] and the dimension reduction algorithm that we propose seems to be useful in designing deconvolution methods.

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1. Introduction

Let $M_{p \times q}(\mathbb{R})$ denote the set of all $p \times q$ matrices with real entries. Consider in a given finite interval $0 \leq t \leq T$ the following linear time-invariant system:

$$\dot{x} = Ax + Bu, \quad (1)$$

$$y = Hx, \quad (2)$$

where $A \in M_{n \times n}(\mathbb{R})$, $B \in M_{n \times d}(\mathbb{R})$, $H \in M_{m \times n}(\mathbb{R})$, $u = u(t)$ is a control policy (input) taking values from \mathbb{R}^d , $x = x(t) \in \mathbb{R}^n$ denotes the state of the system, and $y = y(t) \in \mathbb{R}^m$ is the output trajectory.

By the variation of constants formula we can see that for every initial state $x_0 \in \mathbb{R}^n$ and control (input) $u \in L^1(0, T; \mathbb{R}^d)$ the corresponding output is given by

$$y(t) = He^{At}x_0 + \int_0^t He^{A(t-s)}Bu(s) ds, \quad \forall t \in [0, T]. \quad (3)$$

Let $\mathcal{AC}([0, T]; \mathbb{R}^m)$ denote the space of absolutely continuous functions on $[0, T]$ with values in \mathbb{R}^m . Let $Q : L^1(0, T; \mathbb{R}^d) \rightarrow \mathcal{AC}([0, T]; \mathbb{R}^m)$ be the operator defined by

$$(Qu)(t) = \int_0^t He^{A(t-s)}Bu(s) ds, \quad \forall t \in [0, T]. \quad (4)$$

Obviously, the range of Q , denoted $\text{Range } Q$, does not cover the whole $\mathcal{AC}([0, T]; \mathbb{R}^m)$ (in particular, every function from $\text{Range } Q$ vanishes at $t = 0$).

We continue with the following definition related to system (1) and (2) (see, e.g., [1, p. 167]):

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Definition. The system input is said to be *identifiable (detectable)* if for every initial state x_0 and output $y = y(t)$ the corresponding input $u = u(t)$ is *unique* (u is supposed to exist as long as an output y is produced).

Remark 1. If u is unique for some x_0, y , then the same property holds for all inputs u . In fact, *input identifiability (or detectability)* for system (1) and (2) means that the kernel of Q is the null space: $\ker Q = \{0\}$. If the system input is identifiable, then the system is said to be *left invertible* or *ideally observable* in the Russian literature (see [2]).

Remark 2. If system (1) and (2) is left invertible (i.e., equivalently, $\ker Q = \{0\}$), then the following rank condition holds (see [1, p. 168]):

$$\text{Rank}\{B^T H^T, B^T A^T H^T, \dots, B^T (A^T)^{n-1} H^T\} = d,$$

where the superscript T denotes the matrix transpose. The converse implication is not true (i.e., the above rank condition is not sufficient for left invertibility), as the following simple counterexample shows: $A =$ the matrix with rows $(-1, 0, 0), (0, 0, 0), (0, 0, 1), B =$ the matrix with rows $(1, 0), (0, 1), (0, 0), H = (1, 1, 0)$. Obviously, the rank condition is satisfied, but $\ker Q$ contains the nonzero function $u(t) = \text{col}(t, -1 + e^{-t})$. Therefore, Theorem 5.5.2 in [1, p. 167] is false.

2. The main result

In the previous section we have seen that the problem of left invertibility for system (1) and (2) reduces to the condition $\ker Q = \{0\}$. In this section we formulate a necessary and sufficient condition on matrices A, B, H such that $\ker Q = \{0\}$. Our result relies on an iterative process. Namely, we construct iteratively a non-increasing sequence of integers $\{d_i\} \subset \mathbb{N}$ as well as sequences of matrices $\{A_i\} \subset M_{n \times n}(\mathbb{R}), \{B_i\} \subset M_{n \times d_i}(\mathbb{R})$ and $\{H_i\} \subset M_{m \times n}(\mathbb{R})$ for $i = 0, 1, 2, \dots$ as follows.

Let $A_0 = A, B_0 = B, H_0 = H$ and $d_0 = d$. Given A_i, B_i, H_i and d_i , let $\dim \ker(H_i B_i) =: d_{i+1}$. If $d_{i+1} = 0$, the iterations terminate. Let $d_{i+1} > 0$. Then, $\dim \text{Range}(H_i B_i) = d_i - d_{i+1}$. Moreover, if $\mathbb{R}^{d_i} = U_i \oplus \ker H_i B_i$ and $\mathbb{R}^m = V_i \oplus \text{Range}(H_i B_i)$ (e.g., $U_i := (\ker H_i B_i)^\perp, V_i := (\text{Range}(H_i B_i))^\perp$), then $U_i \cong \text{Range}(H_i B_i) \cong \mathbb{R}^{d_i - d_{i+1}}$.

First, assume that $d_i > d_{i+1}$. Then, there are matrices $M_i \in M_{d_i \times (d_i - d_{i+1})}(\mathbb{R}), T_i \in M_{d_i \times d_{i+1}}(\mathbb{R}), C_i \in M_{(d_i - d_{i+1}) \times m}(\mathbb{R})$, such that

$$\begin{aligned} \text{Range}(M_i) &= U_i, & \text{Range}(T_i) &= \ker H_i B_i, \\ C_i H_i B_i M_i &= \text{id}|_{\mathbb{R}^{d_i - d_{i+1}}} \end{aligned}$$

and for any $x \in \mathbb{R}^{d_i}$ there exist vectors $y \in \mathbb{R}^{d_i - d_{i+1}}, z \in \mathbb{R}^{d_{i+1}}$, uniquely determined, such that $x = M_i y + T_i z$. Let P_i be the matrix of the projection on V_i with respect to $\text{Range}(H_i B_i)$. The matrix C_i may be chosen as

$$C_i = (M_i^T B_i^T H_i^T H_i B_i M_i)^{-1} M_i^T B_i^T H_i^T.$$

Indeed, the matrix $M_i^T B_i^T H_i^T H_i B_i M_i$ is the Gramm matrix of the vectors defined by the columns of $H_i B_i M_i$ and due to the fact that they are linearly independent C_i is well-defined. Note that the matrices M_i, T_i, C_i and P_i are not uniquely determined.

We define the matrices A_{i+1}, B_{i+1} and H_{i+1} by

$$A_{i+1} := (I - B_i M_i C_i H_i) A_i, \quad B_{i+1} := B_i T_i, \quad H_{i+1} := P_i H_i A_i.$$

Now, assume that $d_{i+1} = d_i$. In this case we define

$$A_{i+1} := A_i, \quad B_{i+1} := B_i, \quad H_{i+1} := H_i A_i.$$

Let us now state our main result:

Theorem 1. Let $A \in M_{n \times n}(\mathbb{R}), B \in M_{n \times d}(\mathbb{R}), H \in M_{m \times n}(\mathbb{R})$, and let Q be the operator defined before by (4). If $\{d_i\} \subset \mathbb{N}, \{A_i\} \subset M_{n \times n}(\mathbb{R}), \{B_i\} \subset M_{n \times d_i}(\mathbb{R})$ and $\{H_i\} \subset M_{m \times n}(\mathbb{R})$ are the sequences defined above, then $\ker Q = \{0\}$ if and only if there exists $i \in \mathbb{N}, i \leq dn$ such that $d_i = 0$. Moreover, if $d_i = d_{i+1} = \dots = d_{i+n-1} > 0$ for some i , then $d_k = d_i$ for all $k \geq i$.

Proof. Let $w \in \ker Q$, i.e., $w \in L^1(0, T; \mathbb{R}^d)$ and

$$\int_0^t H e^{A(t-s)} B w(s) ds = 0 \quad \forall t \in [0, T]. \tag{5}$$

By differentiation with respect to t we see that for a.a. $t \in [0, T]$

$$H B w(t) + \int_0^t H A e^{A(t-s)} B w(s) ds = 0. \tag{6}$$

First, assume that $d_1 = \dim \ker(HB) < d = d_0$. If $d_1 = 0$, then there exists a matrix C such that $CHB = \text{id}|_{\mathbb{R}^d}$ which allows us to write

$$w(t) + C \int_0^t H A e^{A(t-s)} B w(s) ds = 0. \tag{7}$$

Now, Gronwall's lemma implies that w is the null function. Let $d_1 > 0$ and $w(t) = M_0v(t) + T_0w_1(t)$. Then Eq. (6) is equivalent to the system

$$v(t) + C_0 \int_0^t H_0 A_0 e^{A_0(t-s)} B_0 (M_0 v(s) + T_0 w_1(s)) ds = 0, \tag{8}$$

$$P_0 \int_0^t H_0 A_0 e^{A_0(t-s)} B_0 (M_0 v(s) + T_0 w_1(s)) ds = 0. \tag{9}$$

It follows by standard arguments that for each integrable function $w_1(t)$ there is a unique solution $v(t)$ of Eq. (8). Moreover, v can be expressed as a convolution product of a suitable matrix kernel and w_1 (see (12)). Indeed, let us define

$$V(t) := \int_0^t e^{A_0(t-s)} B_0 (M_0 v(s) + T_0 w_1(s)) ds. \tag{10}$$

We have for a.a. $t \in [0, T]$

$$\dot{V}(t) = A_0 V(t) + B_0 M_0 v(t) + B_0 T_0 w_1(t),$$

which implies (see (8))

$$\dot{V}(t) = (A_0 - B_0 M_0 C_0 H_0 A_0) V(t) + B_0 T_0 w_1(t).$$

Since $V(0)$ is the null matrix, we obtain by the variation of constants formula

$$V(t) = \int_0^t e^{(A_0 - B_0 M_0 C_0 H_0 A_0)(t-s)} B_0 T_0 w_1(s) ds. \tag{11}$$

Thus

$$v(t) = -C_0 H_0 A_0 \int_0^t e^{(A_0 - B_0 M_0 C_0 H_0 A_0)(t-s)} B_0 T_0 w_1(s) ds \tag{12}$$

is the (unique) solution of Eq. (8) corresponding to w_1 . For this v , according to (10) and (11), we can write

$$P_0 \int_0^t H_0 A_0 e^{A_0(t-s)} B_0 (M_0 v(s) + T_0 w_1(s)) ds = P_0 H_0 A_0 \int_0^t e^{(A_0 - B_0 M_0 C_0 H_0 A_0)(t-s)} B_0 T_0 w_1(s) ds.$$

Thus it is obvious that the existence of a nonzero solution $w(t)$ of Eq. (5) (which is equivalent to system (8) and (9)) is equivalent to the existence of a nonzero solution $w_1(t)$ of the equation

$$\int_0^t H_1 e^{A_1(t-s)} B_1 w_1(s) ds = 0, \quad w_1(t) \in \mathbb{R}^{d_1},$$

where

$$H_1 := P_0 H_0 A_0, \quad A_1 := (I - B_0 M_0 C_0 H_0) A_0, \quad B_1 := B_0 T_0.$$

Next, assume that $d_1 = \dim \ker(HB) = d = d_0$. Then Eq. (6) reads

$$\int_0^t H A e^{A(t-s)} B w(s) ds = 0, \tag{13}$$

so we can take $w_1 = w$,

$$A_1 := A_0, \quad B_1 := B_0, \quad H_1 := H_0 A_0,$$

and we can continue further to construct iteratively matrices A_i, B_i and H_i . Notice that if for some integer i we have $d_i = d_{i+1} = \dots = d_{i+n-1}$, then

$$H_i B_i = H_i A_i B_i = \dots = H_i A_i^{n-1} B_i = \mathcal{O},$$

where \mathcal{O} denotes the zero matrix in $M_{m \times d_i}(\mathbb{R})$. Thus the Cayley–Hamilton theorem implies that $H_i A_i^k B_i = \mathcal{O}$ for any integer $k \geq i$. In particular, it follows that $H_i e^{A_i t} B_i = \mathcal{O}$ and there exists a nonzero function $w(t)$ satisfying (5). Moreover, either there exists an integer $i \leq dn$, such that $d_i = 0$ and the iterations terminate or $0 < d_{dn} = d_{dn+1} = \dots = d_{dn+k} = \dots$ and there exists a nonzero solution of Eq. (5). \square

3. Concluding comments

First of all, it has been brought to our attention that our necessary and sufficient condition formulated in [Theorem 1](#) is a variation of the classical condition described in Property 4.3.6 of Basile and Marro [2, Chapter 4], which involves the geometrical concept of controlled invariant. Indeed, our condition is equivalent to the classical one. This can be shown by comparing the two corresponding algorithms, under the maximal rank condition on B . We do not assume in [Theorem 1](#) that B has maximal rank (i.e., equivalently, $\ker B = \{0\}$), but obviously it is a necessary condition for left invertibility. This equivalence confirms the validity of our result.

While the classical approach is geometrical, our new iterative process relies on simple arguments from linear algebra and the theory of differential equations which allow dimension reduction, as described in the proof of [Theorem 1](#).

If the system is left invertible (i.e., $\ker Q = \{0\}$), one can use an iterative process suggested by the proof of [Theorem 1](#) to solve for $u = u(t)$ the equation $Qu(t) = f(t)$. This operation is nowadays called *deconvolution* since Q is an integral convolution operator. In general u does not depend continuously on f and this makes the problem difficult. Among the existing papers addressing deconvolution methods, we refer the reader to [3–5] and the references therein. We think that our iterative process could generate new efficient deconvolution methods.

The general output equation $y(t) = Hx(t) + Du(t)$, $t \in [0, T]$, can also be considered in our framework. Here D is an $m \times d$ matrix with real entries. In this case, instead of Eq. (5), we have the following integral equation:

$$Dw(t) + H \int_0^t e^{A(t-s)} Bw(s) ds = 0,$$

whose form is similar to Eq. (6). Therefore, one can apply our algorithm described in the proof of [Theorem 1](#) above to derive a necessary and sufficient condition for input identification (or left invertibility). The precise formulation of this condition is left to the reader. In particular, if $\ker D = \{0\}$, then obviously the system input is identifiable.

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