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Optimal Placement of a Deposit between Markets: Riemann-Finsler Geometrical Approach

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Abstract By using the Riemann-Finsler geometry, we study the existence and location of the optimal points for a general cost function involving Finsler distances. Our minimization problem provides a model for the placement of a deposit within a domain with several markets such that the total transportation cost is minimal. Several concrete examples are studied either by precise mathematical tools or by evolutionary (computer assisted) techniques.

Keywords General cost functions · Riemann-Finsler manifolds · Optimal deposit placements · Evolutionary techniques

1 Introduction and Motivation

Various practical problems in economics lead to minimization problems. Typical cases occur when numerical data are adjusted by means of the least square method or optimal (equilibrium) points are found for certain cost functions, etc.

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The purpose of the present paper is to treat a natural economical question: *There are $n \in \mathbb{N}$ markets and their coordinates are known. Find the possible places for those deposits from which* (*resp. to which*) *the sum of certain transportation costs to* (*resp. from*) *markets attains the minimum*. Here, the costs depend on the distances between the deposit and the given n markets.

At first glance, the problem looks very simple; however, serious technical problems may occur even in simple situations.[1] To show this, let us consider three markets P_1, P_2, P_3 placed on an inclined plane (slope) with an angle α to the horizontal plane, denoted by (S_α). Assume that three cars transport products from (resp. to) deposit $P \in (S_\alpha)$ to (resp. from) markets $P_1, P_2, P_3 \in (S_\alpha)$ such that:

- they move always in (S_α) along straight roads;
- the Earth gravity acts on them (we omit other physical perturbations such as friction, air resistance, etc.);
- the transport costs coincide with the *distance* (measuring actually the *time* elapsed to arrive) from (resp. to) deposit P to (resp. from) markets P_i $(i = 1, 2, 3)$.

We emphasize that usually the two distances, i.e., from the deposit to the markets and conversely, are *not* the same. The point here is that the travel speed depends heavily on both the slope of the terrain and the direction of travel. More precisely, if a car moves with a constant speed v [m/s] on a horizontal plane, it goes $l_t = vt + \frac{g}{2}t^2 \sin\alpha \cos\theta$ meters in t seconds on (S_α), where θ is the angle between the straight road and the direct downhill road (θ is measured in clockwise direction). The law of the above phenomenon can be described relatively to the horizontal plane by means of the parametrized function

$$F_\alpha(y_1, y_2) = \frac{y_1^2 + y_2^2}{v\sqrt{y_1^2 + y_2^2} + \frac{g}{2} y_1 \sin\alpha}, \quad (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{1}$$

Here, $g \approx 9.81 \frac{m}{s^2}$. The *distance* (measuring the *time* to arrive) from $P = (P^1, P^2)$ to $P_i = (P_i^1, P_i^2)$ is

$$d_\alpha(P, P_i) = F_\alpha(P_i^1 - P^1, P_i^2 - P^2),$$

and for the converse it is

$$d_\alpha(P_i, P) = F_\alpha(P^1 - P_i^1, P^2 - P_i^2).$$

Consequently, we have to minimize the functions

$$C_f(P) = \sum_{i=1}^{3} d_\alpha(P, P_i) \quad \text{and} \quad C_b(P) = \sum_{i=1}^{3} d_\alpha(P_i, P), \tag{2}$$

[1] A mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts. It should be to us a guide post on the mazy paths to hidden truths, and ultimately a reminder of our pleasures in the successful solution. (D. Hilbert)

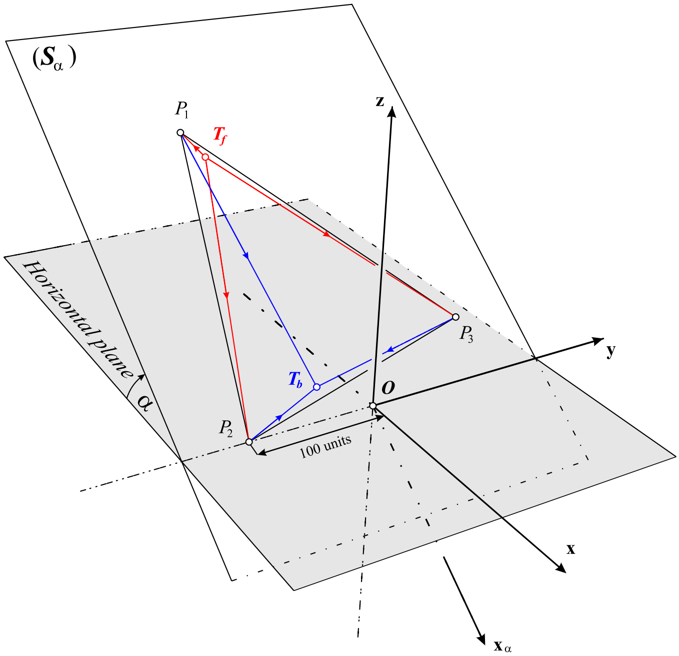

Fig. 1 We fix $P_1 = (-250, -50)$, $P_2 = (0, -100)$ and $P_3 = (-50, 100)$ on the slope (S_α) with angle $\alpha = 35°$. If $v = 10$, the minimum of the total forward cost on the slope is $C_f \approx 40.3265$; the corresponding deposit is located at $T_f \approx (-226.11, -39.4995) \in (S_\alpha)$. However, the minimum of the total backward cost on the slope is $C_b \approx 38.4143$; the corresponding deposit has the coordinates $T_b \approx (-25.1332, -35.097) \in (S_\alpha)$. Numerical data are provided by 2 000 iterations of search operators involved in the genetic algorithm described in Sect. 2.3

when P moves on (S_α). The function C_f (resp. C_b) denotes the *total forward* (*resp. backward*) *cost* between the deposit $P \in (S_\alpha)$ and markets $P_1, P_2, P_3 \in (S_\alpha)$. The minimum points of C_f and C_b, respectively, may be far from each other (see Fig. 1), due to the fact that F_α (and d_α) is not symmetric unless $\alpha = 0$, i.e., $F_\alpha(-y_1, -y_2) \neq F_\alpha(y_1, y_2)$ for each $(y_1, y_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. We will use in general T_f (resp. T_b) to denote a minimum point of C_f (resp. C_b), which corresponds to the position of a deposit when we measure costs in forward (resp. backward) manner, see (2).

In the case $\alpha = 0$ (when (S_α) is a horizontal plane), the functions C_f and C_b coincide (the same is true for T_f and T_b). The minimum point $T = T_f = T_b$ is the well-known *Torricelli point* corresponding to the triangle $P_1P_2P_{3\Delta}$. Note that $F_0(y_1, y_2) = \sqrt{y_1^2 + y_2^2}/v$ corresponds to the standard Euclidean metric; indeed,

$$d_0(P, P_i) = d_0(P_i, P) = \sqrt{(P_i^1 - P^1)^2 + (P_i^2 - P^2)^2}/v$$

measures the time, which is needed to arrive from P to P_i (and vice-versa) with constant velocity v. See Fig. 1.

Unfortunately, finding critical points as possible minima does not yield any result: either the minimization function is not smooth enough (usually, it is only a locally

Lipschitz function) or the system, which would give the critical points, becomes very complicated even in quite simple cases (see (9) below). Consequently, the main purpose of the present paper is to study the set of these minima (existence, location) in various geometrical settings.

Note, that the function appearing in (1) is a typically Finsler metric on $\mathbb{R}^2$, introduced and studied first by Matsumoto in [1]; see also [2]. In this way, elements from Riemann-Finsler geometry are needed in order to handle the question formulated above. Thus, in the first part of Sect. 2, we recall some basic notions from Riemann-Finsler geometry and we formulate the main question of this paper in its full generality. Then, some facts on the evolutionary approach (genetic algorithm) are recalled, which are extremely useful for numerical calculations; by using this technique, we are able to localize/approximate the global minima of the total cost functions. In Sect. 3, we prove some existence, uniqueness and multiplicity results. Relevant numerical (counter)examples are constructed by means of evolutionary methods and computational geometry tools, emphasizing the applicability and sharpness of our results.

2 Formulation of the Main Problem. Mathematical and Evolutionary Approach

In this section, we recall briefly basic notions and results from Riemann-Finsler geometry and evolutionary approach, which will be used in the sequel. In Sect. 2.2, we give the precise formulation of the main problem.

2.1 Elements from Riemann-Finsler Geometry

Let M be a connected m-dimensional C^∞ manifold and let $TM = \bigcup_{p\in M} T_pM$ be its tangent bundle. If the continuous function $F : TM \to [0, \infty)$ satisfies the conditions that it is C^∞ on $TM \setminus \{0\}$; $F(tu) = tF(u)$ for all $t \geq 0$ and $u \in TM$, i.e., F is positively homogeneous of degree one; and the matrix $g_{ij}(u) := (\frac{1}{2}F^2)_{y^i y^j}(u)$ is positive definite for all $u \in TM \setminus \{0\}$, then we say that (M, F) is a *Finsler manifold*.

Let $\gamma : [0, r] \to M$ be a piecewise C^∞ curve. Its *integral length* is defined as

$$L(\gamma) = \int_0^r F(\gamma(t), \dot{\gamma}(t))\, dt.$$

For $x_0, x_1 \in M$, denote by $\Gamma(x_0, x_1)$ the set of all piecewise C^∞ curves $\gamma : [0, r] \to M$ such that $\gamma(0) = x_0$ and $\gamma(r) = x_1$. Define a map $d_F : M \times M \to [0, \infty)$ by

$$d_F(x_0, x_1) = \inf_{\gamma\in\Gamma(x_0,x_1)} L(\gamma). \tag{3}$$

Of course, we have

$$d_F(x_0, x_1) \geq 0,$$

where equality holds if and only if $x_0 = x_1$, and

$$d_F(x_0, x_2) \leq d_F(x_0, x_1) + d_F(x_1, x_2),$$

the so-called *triangle inequality*. In general, since F is only a positive homogeneous function, $d_F(x_0, x_1) \neq d_F(x_1, x_0)$; therefore, (M, d_F) is only a nonreversible metric space.

Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM \setminus \{0\} \to M$. Unlike the Levi-Civita connection in Riemann geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the *Chern connection* whose coefficients are denoted by Γ^k_{ij}; see p. 38 in [2]. This connection induces the *curvature tensor*, denoted by R; see Chap. 3 in [2].

A C^∞ curve $\sigma : [0, 1] \to M$ is said to be a constant Finslerian speed *geodesic* if

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}(\Gamma^i_{jk})_{(\sigma,T)} = 0, \quad i = 1, \dots, m = \dim M, \tag{4}$$

where T denotes the velocity field associated with σ. A Finsler manifold (M, F) is said to be *geodesically complete* if every geodesic $\sigma : [0, 1] \to M$ parameterized to have constant Finslerian speed, can be extended to a geodesic defined on $(-\infty, \infty)$.

A set $M_0 \subseteq M$ is *forward bounded* if there exist $p \in M$ and $r > 0$ such that $M_0 \subseteq \{x \in M : d_F(p, x) < r\}$. Similarly, $M_0 \subseteq M$ is *backward bounded* if there exist $p \in M$ and $r > 0$ such that $M_0 \subseteq \{x \in M : d_F(x, p) < r\}$.

Let $(x, y) \in TM \setminus 0$ and let V be a section of the pulled-back bundle π^*TM. Then,

$$K(y, V) = \frac{g_{(x,y)}(R(V, y)y, V)}{g_{(x,y)}(y, y)g_{(x,y)}(V, V) - [g_{(x,y)}(y, V)]^2} \tag{5}$$

is the *flag curvature* with flag y and transverse edge V. Here,

$$g_{(x,y)} := g_{ij(x,y)}dx^i \otimes dx^j := \left(\frac{1}{2}F^2\right)_{y^i y^j} dx^i \otimes dx^j$$

is the Riemannian metric on the pulled-back bundle π^*TM; see p. 68 in [2]. When F is Riemannian, then the flag curvature coincides with the sectional curvature. Let K be the collection of flag curvatures $\{K(V, W) : 0 \neq V, W \in T_xM$, $x \in M$; V and W are not collinear$\}$. We say that the flag curvature of (M, F) is *non-positive* if $K \leq 0$.

A Finsler manifold is of *Berwald type* if the Chern connection coefficients Γ^k_{ij} in natural coordinates depend only on the base point. Special Berwald spaces are the (*locally*) *Minkowski spaces* and the *Riemann manifolds*. In the latter case, the Chern connection coefficients Γ^k_{ij} coincide the Christofel symbols.

2.2 Formulation of the Main Problem

Let (M, F) be a connected Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one and let d_F be from (3). Consider the points $P_i \in M$, $i = 1, \dots, n$, corresponding to the markets. We are looking for the *set of minima* of the cost functions $C_f(P_i, n, s), C_b(P_i, n, s) : M \to [0, \infty)$, corresponding to the place of the deposit(s), defined by

$$C_f(P_i, n, s)(P) = \sum_{i=1}^{n} d_F^s(P, P_i) \quad \text{and} \quad C_b(P_i, n, s)(P) = \sum_{i=1}^{n} d_F^s(P_i, P),$$

where $s \geq 1$. The value $C_f(P_i, n, s)(P)$ (resp. $C_b(P_i, n, s)(P)$) denotes the *total s-forward* (*resp. s-backward*) *cost* between the deposit $P \in M$ and the markets $P_i \in M$, $i = 1, \ldots, n$. When $s = 1$, we simply say *total forward* (*resp. backward*) *cost*.

2.3 Evolutionary Approach

Evolutionary techniques are useful and efficient in case of such optimization problems, when the search space is large, complex, or traditional search and numerical methods fail. They are based on the principles of evolution via natural *selection,* employing a population of individuals that undergoes selection in the presence of variation-inducing *operators* such as *mutation* and *recombination.* In the present case, we are interested in finding solutions for the following optimization problems:

$$\min C_f(P_i, n, s)(p), \quad \text{s.t.} \quad p \in M, \tag{6}$$

$$\min C_b(P_i, n, s)(p), \quad \text{s.t.} \quad p \in M. \tag{6'}$$

Equilibrium points and their orbits presented in all figures of the present paper are generated by a simple genetic algorithm. In the sequel, we give a short description of the elements and search operators involved in this evolutionary technique.

2.3.1 Evolutionary Representation

A search process starts with a randomly generated initial *population* $\tilde{\mathrm{P}}_0$ containing the *individuals* $p_1, p_2, \ldots, p_k \in M$. Each individual is determined by *genes*, i.e., by coordinates of points p_i, and each of them has a *fitness value.* In the case of the cost function $C_f(P_i, n, s)$ (resp. $C_b(P_i, n, s)$), the fitness of an individual $p_i \in \tilde{\mathrm{P}}_l$, $l \geq 0$, is given by

$$\operatorname{eval}(p_i) = -C_f(P_i, n, s)(p_i) \qquad (\text{resp. } \operatorname{eval}(p_i) = -C_b(P_i, n, s)(p_i)).$$

If $\operatorname{eval}(p_i) > \operatorname{eval}(p_j)$ $(i \neq j,\ i, j \in \{1, 2, \ldots, k\})$, the individual p_i is considered *better* than the individual p_j.

2.3.2 Search Operators

Selection and mutation operators are presented below. During the evolution process, recombination (crossover) is not used. If someone is looking for solutions of optimization problems (6) in a Finsler manifold in which the parametric form of geodesics is unknown, then the genetic representation of individuals can be changed by using some type of curve interpolation or approximation method (e.g., by control polygons and knot vectors which determine cubic splines or cubic nonuniform rational B-splines, see [3]); in this case, one can define also a recombination operator, e.g., switching the parts of control polygons of two individuals selected at random.

2.3.3 Selection

Tournament selection was used for determining the individuals of the next generations. This operator runs a "tournament" among a few individuals chosen at random from the actual population $\tilde{\mathrm{P}}_l,\ l \geq 0$, and selects the winner (the one with the best fitness) for mutation. Repeating the selection process k times, one gets a new and potentially better generation $\tilde{\mathrm{P}}_{l+1}$ than population $\tilde{\mathrm{P}}_l$.

2.3.4 Mutation

Mutation is done by perturbing the coordinates of a randomly selected individual $p_i \in \tilde{\mathrm{P}}_l,\ i \in \{1, 2, \ldots, k\},\ l \geq 0$ such that $p_i \in M$. Mutation occurs during evolution according to a user-definable mutation probability $p_{\mathrm{mut}} \in (0, 1]$.

3 Results and Examples

In this section we prove some mathematical results concerning the set of minima for functions $C_f(P_i, n, s)$ and $C_b(P_i, n, s)$.

Let (M, F) be an m-dimensional connected Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one. By using the triangle inequality, for every $x_0, x_1, x_2 \in M$ we have

$$|d_F(x_1, x_0) - d_F(x_2, x_0)| \leq \max\{d_F(x_1, x_2), d_F(x_2, x_1)\}. \tag{7}$$

Given any point $P \in M$, there exists a coordinate map φ_P defined on the closure of some precompact open subset U containing P such that φ_P maps the set U diffeomorphically onto the open Euclidean ball $B^m(r)$, $r > 0$, with $\varphi_P(P) = 0_{\mathbb{R}^m}$. Moreover, there is a constant $c > 1$, depending on only P and U such that

$$c^{-1}\|\varphi_P(x_1) - \varphi_P(x_2)\| \leq d_F(x_1, x_2) \leq c\|\varphi_P(x_1) - \varphi_P(x_2)\|, \tag{8}$$

for every $x_1, x_2 \in U$; see p. 149 in [2]. Here, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$. We claim that for every $Q \in M$, the function $d_F(\varphi_P^{-1}(\cdot), Q)$ is a Lipschitz function on $\varphi_P(U) = B^m(r)$. Indeed, for every $y_i = \varphi_P(x_i) \in \varphi_P(U)$, $i = 1, 2$, due to (7) and (8), one has

$$\begin{aligned}|d_F(\varphi_P^{-1}(y_1), Q) - d_F(\varphi_P^{-1}(y_2), Q)| &= |d_F(x_1, Q) - d_F(x_2, Q)| \\ &\leq \max\{d_F(x_1, x_2), d_F(x_2, x_1)\} \leq c\|y_1 - y_2\|.\end{aligned}$$

Consequently, for every $Q \in M$, there exists a *generalized gradient* of the (locally) Lipschitz function $d_F(\varphi_P^{-1}(\cdot), Q)$ on $\varphi_P(U) = B^m(r)$, see p. 27 in [4]; i.e., for every $y \in \varphi_P(U) = B^m(r)$, we have

$$\partial d_F(\varphi_P^{-1}(\cdot), Q)(y) = \{\xi \in \mathbb{R}^m : d_F^0(\varphi_P^{-1}(\cdot), Q)(y; h) \geq \langle \xi, h\rangle \text{ for all } h \in \mathbb{R}^m\},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^m$ and

$$d_F^0(\varphi_P^{-1}(\cdot),Q)(y;h)=\limsup_{z\to y,\ t\to 0^+}\frac{d_F(\varphi_P^{-1}(z+th),Q)-d_F(\varphi_P^{-1}(z),Q)}{t}.$$

Theorem 3.1 (Necessary Condition) *Assume that $T_f\in M$ is a minimum point for $C_f(P_i,n,s)$ and that φ_{T_f} is a map as above. Then,*

$$0_{\mathbb{R}^m}\in\sum_{i=1}^n d_F^{s-1}(T_f,P_i)\partial d_F(\varphi_{T_f}^{-1}(\cdot),P_i)(\varphi_{T_f}(T_f)). \tag{9}$$

Proof Since $T_f\in M$ is a minimum point of the locally Lipschitz function $C_f(P_i,n,s)$, then

$$0_{\mathbb{R}^m}\in\partial\left(\sum_{i=1}^n d_F^s(\varphi_{T_f}^{-1}(\cdot),P_i)\right)(\varphi_{T_f}(T_f));$$

see Proposition 2.3.2 in [4]. Now, using the basic properties of the generalized gradient, see Proposition 2.3.3 and Theorem 2.3.10 in [4], we conclude the proof. □

Remark 3.1 A result similar to Theorem 3.1 can also be obtained for $C_b(P_i,n,s)$.

Example 3.1 Let $M=\mathbb{R}^m$, $m\geq 2$, be endowed with the natural Euclidean metric. Taking into account (9), a simple computation shows that the unique minimum point $T_f=T_b$ (i.e., the place of the deposit) for $C_f(P_i,n,2)=C_b(P_i,n,2)$ is the centre of gravity of markets $\{P_1,\dots,P_n\}$, i.e., $\frac{1}{n}\sum_{i=1}^n P_i$. In this case, φ_{T_f} can be the identity map on $\mathbb{R}^m$.

Remark 3.2 The system (9) may become very complicated even for simple cases; it is enough to consider the Matsumoto metric given by (1). In such cases, we are not able to give an explicit formula for minimal points. This observation leads us to a theoretical study of problem (6).

The next result gives an alternative concerning the number of minimum points of the function $C_f(P_i,n,s)$ in a general geometrical framework. (Similar result can be obtained for $C_b(P_i,n,s)$.) Namely, we have the following theorem.

Theorem 3.2 *Let (M,F) be a simply connected, geodesically complete Berwald manifold of nonpositive flag curvature, where F is positively (but perhaps not absolutely) homogeneous of degree one. Then*:

(a) *there exists either a unique or infinitely many minimum points for $C_f(P_i,n,1)$;*
(b) *there exists a unique minimum point for $C_f(P_i,n,s)$ whenever $s>1$.*

Proof First of all, we observe that M is not a backward bounded set. Indeed, if we assume that it is, then M is compact due to Hopf-Rinow theorem, see p. 172 and

p. 168 in [2]. On the other hand, due to the Cartan-Hadamard theorem, see p. 238 in [2], the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$. Thus, the tangent space $T_pM = \exp_p^{-1}(M)$ is compact, a contradiction. Since M is not backward bounded, in particular, for every $i = 1, \dots, n$, we have that

$$\sup_{P \in M} d_F(P, P_i) = \infty.$$

Consequently, outside of a large backward bounded subset of M, denoted by M_0, the value of $C_f(P_i, n, s)$ is large. But, M_0 being compact, the continuous function $C_f(P_i, n, s)$ attains its infimum, i.e., the set of the minima for $C_f(P_i, n, s)$ is always nonempty.

In [5], the authors prove that, under the above hypotheses, the (nonreversible) metric space (M, d_F) is a Busemann NPC space; i.e., in small geodesic triangles, the Finslerian-length of a side is at least twice the geodesic distance of the midpoints of the other two sides. Due to Corollary 2.2.4 in [6], (M, d_F) is a global Busemann NPC space (i.e., the above property is valid for arbitrarily geodesic triangle). Consequently, for every nonconstant geodesic $\sigma : [0, 1] \to M$ and $p \in M$, the function $t \mapsto d_F(\sigma(t), p)$ is convex (see Corollary 2.2.2 in [6]) and $t \mapsto d_F^s(\sigma(t), p)$ is strictly convex, whenever $s > 1$ (see Corollary 2.2.6 in [6]).

(a) Let us assume that there are at least two minimum points for $C_f(P_i, n, 1)$, denoting them by T_f^0 and T_f^1. Let $\sigma : [0, 1] \to M$ be a geodesic with constant Finslerian speed such that $\sigma(0) = T_f^0$ and $\sigma(1) = T_f^1$. Then, for every $t \in (0, 1)$, we have

$$\begin{aligned} C_f(P_i, n, 1)(\sigma(t)) &= \sum_{i=1}^{n} d_F(\sigma(t), P_i) \\ &\le (1-t)\sum_{i=1}^{n} d_F(\sigma(0), P_i) + t\sum_{i=1}^{n} d_F(\sigma(1), P_i) \\ &= (1-t)\min C_f(P_i, n, 1) + t\min C_f(P_i, n, 1) \\ &= \min C_f(P_i, n, 1). \end{aligned} \tag{10}$$

Consequently, for every $t \in [0, 1]$, $\sigma(t) \in M$ is a minimum point for $C_f(P_i, n, 1)$.

(b) It follows directly from the strict convexity of the function $t \mapsto d_F^s(\sigma(t), p)$, when $s > 1$; indeed, in (10) we have $<$ instead of $\le$ which shows that we cannot have more then one minimum point for $C_f(P_i, n, 1)$.

□

Example 3.2 Let F be the Finsler metric introduced in (1). One can see that $(\mathbb{R}^2, F)$ is a typically nonsymmetric Finsler manifold. Actually, it is a (locally) Minkowski space, so a Berwald space as well; its Chern connection vanishes, see p. 384 in [2]. According to (4) and (5), the geodesics are straight lines (hence $(\mathbb{R}^2, F)$ is geodesically complete) and the flag curvature is identically 0. Thus, we can apply Theorem 3.2. For instance, if we consider the points $P_1 = (a, -b) \in \mathbb{R}^2$ and

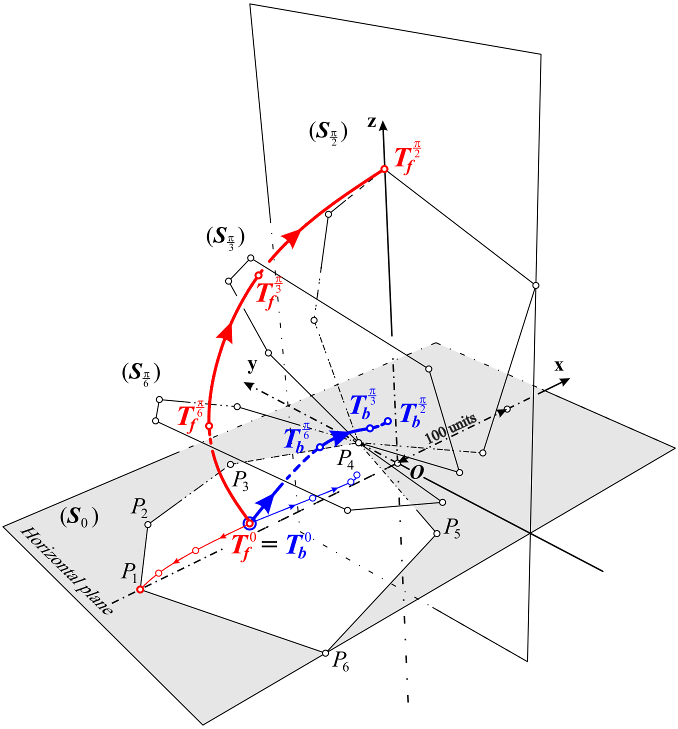

Fig. 2 A hexagon with vertices $P_1, P_2, \dots, P_6$ in the Matsumoto space. Increasing the slope's angle α from 0 to $\pi/2$, points T_f^α and T_b^α are wandering in the presented directions. Orbits of points T_f^α and T_b^α were generated by natural cubic spline curve interpolation

$P_2 = (a, b) \in \mathbb{R}^2$ with $b \neq 0$, the minimum points of the function $C_f(P_i, 2, 1)$ form the segment $[P_1, P_2]$, *independently* of the value of α. The same is true for $C_b(P_i, 2, 1)$. However, considering more complicated constellations, the situation changes dramatically, see Fig. 2. It would be interesting to study in similar cases the precise orbit of the (Torricelli) points T_f^α and T_b^α when α varies from 0 to $\pi/2$. Several numerical experiments show that T_f^α tends to a top point of the convex polygon (as in the Fig. 2).

In the sequel, we want to study our problem in a special constellation: we assume the markets are situated on a common "straight line", i.e., on a geodesic which is in a Riemann manifold. Note that, in the Riemannian context, the forward and backward costs coincide, i.e.,

$$C_f(P_i, n, 1) = C_b(P_i, n, 1).$$

We denote this common value by $C(P_i, n, 1)$. We have the following theorem.

Theorem 3.3 *Let (M, g) be a simply connected, complete Riemann manifold of non-positive sectional curvature. Assume the points $P_i \in M$, $i = 1, \dots, n$, $(n \geq 2)$, belong to a geodesic $\sigma : [0, 1] \to M$ such that $P_i = \sigma(t_i)$ with $0 \leq t_1 < \cdots < t_n \leq 1$. Then:*

(a) *the unique minimum point for* $C(P_i, n, 1)$ *is* $P_{[n/2]}$ *whenever* n *is odd*;
(b) *the minimum points for* $C(P_i, n, 1)$ *are situated on* σ, *between* $P_{n/2}$ *and* $P_{n/2+1}$ *whenever* n *is even*.

In order to prove Theorem 3.3, we recall a well-known result from Riemann geometry.

Lemma 3.1 ([7], Lemma 3.1) *Let* (M, g) *be a simply connected, complete Riemann manifold of nonpositive sectional curvature. Consider the geodesic triangle determined by vertices* $a, b, c \in M$. *If* $\widehat{c}$ *is the angle belonging to vertex* c *and if* $A = d_g(b, c)$, $B = d_g(a, c)$, $C = d_g(a, b)$, *then*

$$A^2 + B^2 - 2AB\cos\widehat{c} \leq C^2,$$

where d_g *denotes the Riemannian metric induced by* $F = g$.

Proof of Theorem 3.3 Since (M, g) is complete, we extend σ to $(-\infty, \infty)$, keeping the same notation. First, we prove that the minimum point(s) for $C(P_i, n, 1)$ belong to the geodesic σ. We assume the contrary, i.e., let $T \in M \setminus \mathrm{Image}(\sigma)$ be a minimum point of $C(P_i, n, 1)$. Let $T_\perp \in \mathrm{Image}(\sigma)$ be the projection of T on the geodesic σ, i.e.

$$d_g(T, T_\perp) = \min_{t \in \mathbb{R}} d_g(T, \sigma(t)).$$

It is clear that the (unique) geodesic lying between T and $T_\perp$ is perpendicular to σ. Now, let $i_0 \in \{1, \ldots, n\}$ such that $P_{i_0} \neq T_\perp$. Applying Lemma 3.1 to the triangle with vertices P_{i_0}, T and $T_\perp$ (so, $\widehat{T_\perp} = \pi/2$), we have

$$d_g^2(T_\perp, T) + d_g^2(T_\perp, P_{i_0}) \leq d_g^2(T, P_{i_0}).$$

Since

$$d_g(T_\perp, T) > 0,$$

we have

$$d_g(T_\perp, P_{i_0}) < d_g(T, P_{i_0}).$$

Consequently,

$$C(P_i, n, 1)(T_\perp) = \sum_{i=1}^{n} d_g(T_\perp, P_i) < \sum_{i=1}^{n} d_g(P, P_i) = \min C(P_i, n, 1),$$

a contradiction.

Now, conclusions (a) and (b) follow easily by using simple arithmetical reasons. □

Theorem 3.3 is sharp in the following sense: neither the nonpositivity of the sectional curvature (see Example 3.3) nor the Riemannian structure (see Example 3.4) can be omitted.

Example 3.3 (Sphere) Let us consider the 2-dimensional unit sphere $S^2 \subset \mathbb{R}^3$ endowed with its natural Riemannian metric g_0 inherited by $\mathbb{R}^3$. We know that it has constant curvature 1. Let us fix $P_1, P_2 \in S^2$ ($P_1 \neq P_2$) and their antipodals $P_3 = -P_1$, $P_4 = -P_2$. There exists a unique great circle (geodesic) connecting P_i, $i = 1, \dots, 4$. However, we observe that the function $C(P_i, 4, 1)$ is *constant* on S^2; its value is 2π. Consequently, *every* point on S^2 is a minimum for the function $C(P_i, 4, 1)$.

Example 3.4 (Finslerian Poincaré disc) Let us consider the disc

$$M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}.$$

Introducing the polar coordinates (r, θ) on M, i.e., $x = r\cos\theta$, $y = r\sin\theta$, we define a Finsler metric

$$F((r,\theta), V) = \frac{1}{1 - \frac{r^2}{4}}\sqrt{p^2 + r^2 q^2} + \frac{pr}{1 - \frac{r^4}{16}},$$

where

$$V = p\frac{\partial}{\partial r} + q\frac{\partial}{\partial \theta} \in T_{(r,\theta)}M.$$

(M, F) is a special (nonsymmetric) Finsler manifold, a so-called *Randers space;* see Sect. 12.6 in [2]:

- it has constant negative flag curvature $-\frac{1}{4}$;
- the geodesics in (M, F) have the following trajectories: Euclidean circular arcs that intersect the boundary of M at Euclidean right angles; Euclidean straight rays that emanate from the origin; and Euclidean straight rays that aim toward the origin.

Although (M, F) is forward geodesically complete (i.e., every geodesic $\sigma : [0, 1] \to M$ can be extended to $[0, \infty)$), it has constant negative flag curvature $-\frac{1}{4}$ and it is contractible (thus, simply connected), the conclusion of Theorem 3.3 may be false. Indeed, one can find points in M (belonging to the same geodesic) such that the minimum point for the total forward (resp. backward) cost function is *not* situated on the geodesic, see Fig. 3.

Note that this example (Finslerian Poincaré disc) may give a model of a gravitational field whose centre of gravity is located at the origin O, while the boundary ∂M means the "infinity". Suppose that in this gravitational field, we have several spaceships, which are delivering some cargo to certain bases or to another spacecraft. Also, assume that these spaceships are of the same type and they consume k liter/second fuel ($k > 0$). Note that the expression $F(d\sigma)$ denotes the physical *time* elapsed to traverse a short portion $d\sigma$ of the spaceship orbit. Consequently, traversing a short path $d\sigma$, a spaceship consumes $kF(d\sigma)$ liter of fuel. In this way, the number $k\int_0^1 F(\sigma(t), d\sigma(t))dt$ expresses the quantity of fuel used up by a spaceship traversing an orbit $\sigma : [0, 1] \to M$.

Suppose that two spaceships have to meet each other (for logistical reasons) starting their trip from bases P_1 and P_2, respectively. Consuming as low total quantity

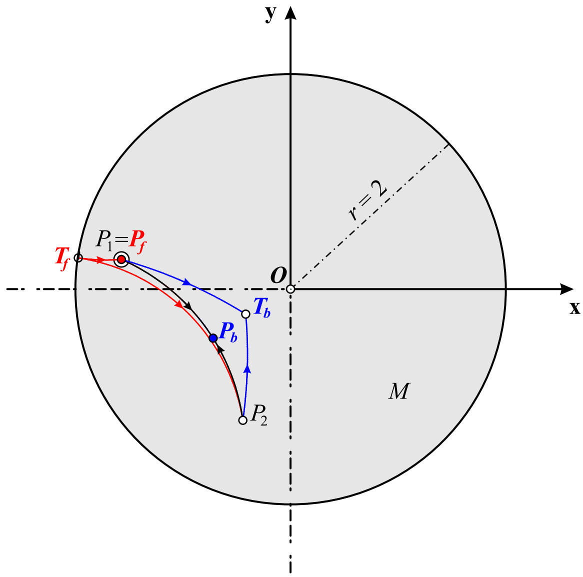

Fig. 3 First, the search space is limited to the geodesic determined by $P_1(1.6, 170°)$ and $P_2(1.3, 250°)$. After 2 000 iterations of the presented genetic algorithm, the point which minimizes the total backward (resp. forward) cost function $C_b(P_i, 2, 1)$ (resp. $C_f(P_i, 2, 1)$) is approximated by $P_b(0.8541, 212.2545°)$ (resp. $P_f = P_1$); in this case $C_b(P_i, 2, 1)(P_b) \approx 1.26$ (resp. $C_f(P_i, 2, 1)(P_f) \approx 2.32507$). However, if the search space is the whole Randers space M, after 2 000 iterations of search operators, the minimum point of total backward (resp. forward) cost function is approximated by $T_b(0.4472, 212.5589°)$ (resp. $T_f(1.9999, 171.5237°)$), which gives $C_b(P_i, 2, 1)(T_b) \approx 0.950825 < C_b(P_i, 2, 1)(P_b)$ (resp. $C_f(P_i, 2, 1)(T_f) \approx 2.32079 < C_f(P_i, 2, 1)(P_f)$)

of fuel as possible, they will choose T_b as a meeting point and *not* P_b on the geodesic determined by P_1 and P_2. Thus, the point T_b could be a position for an optimal deposit-base.

Now, suppose that we have two damaged spacecraft (e.g., without fuel) at positions P_1 and P_2. Two rescue spaceships consuming as low total quantity of fuel as possible, will blastoff from base T_f and not from $P_f = P_1$ on the geodesic determined by P_1 and P_2. In this case, the point T_f is the position for an optimal rescue-base. If the spaceships in trouble are close to the center of the gravitational field M, then any rescue-base located closely also to the center O, implies the consumption of a great amount of energy (fuel) by the rescue spaceships in order to reach their destinations (namely, P_1 and P_2). Indeed, they have to overcome the strong gravitational force near the centre O. Consequently, this is the reason why the point T_f is so far from O, as Fig. 3 shows. Note that further numerical experiments support this observation. However, there are certain special cases when the position of the optimal rescue-base is either P_1 or P_2: from these two points, the farthest one from the gravitational center O will be the position of the rescue-base. In such case, the orbit of the (single) rescue spaceship is exactly the geodesic determined by points P_1 and P_2.

4 Final Remarks

(i) In a forthcoming paper, we plan to develop mathematical tools and evolutionary techniques to prove the existence and location of minimum points of the functions $C_f(P_i, n, s)$ and $C_b(P_i, n, s)$ subjected to certain *constraints* represented as fixed subsets/submanifolds of the initial Finsler manifold (M, F). A useful starting point for this study can be found in [8–10], where the authors formulated some nonlinear programming problems in the language of differential geometry, imposing the constraint set as a finite-dimensional differentiable manifold M. Moreover, as one can see, the proof of Theorem 3.2 is based on the convexity of certain functions, exploiting the nonpositivity of the flag curvature of the Berwald (in particular, locally Minkowski or Riemann) manifold. We think that new kind of results could be obtained (even for problems with constraints) by using more general convexity notions, as introduced in [11].

(ii) We intend to improve the genetic algorithm presented in Sect. 2.3 for Finsler manifolds in which the parametric form of the geodesics is unknown, i.e., we wish to propose a multimodal evolutionary technique based on curve interpolation or approximation methods, which is able to detect the unknown geodesics in such Finsler manifolds.

(iii) An important problem in fluid mechanics is the Monge-Kantorovich mass transfer (shortly, MKMT) problem. In [12], the authors considered the MKMT problem on metric spaces with possibly unbounded cost functions. We believe strongly that our approach/results can be successfully applied to MKMT problems when the phenomenon occurs in a (not necessarily symmetric) Finsler medium.

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