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On a nonsmooth fourth order boundary value problem

Tihomir Gyulov[1], Gheorghe Moroşanu*

Department of Mathematics and Its Applications, Central European University, Nador u. 9, 1051 Budapest, Hungary

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Abstract

Existence of solutions of the following boundary value problem is investigated by a variational approach.

$$u^{iv} - au'' + bu \in \overline{\partial} F(t, u),$$
$$\begin{pmatrix} -u'''(0) + au'(0) \\ u'''(1) - au'(1) \\ u''(0) \\ -u''(1) \end{pmatrix} \in \partial j \begin{pmatrix} u(0) \\ u(1) \\ u'(0) \\ u'(1) \end{pmatrix}.$$

Here, $F(t, \xi) : (0, 1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, locally Lipschitz with respect to the second variable, and $\overline{\partial} F(t, \xi)$ denotes the generalized Clarke gradient of $F(t, \xi)$ with respect to ξ, while j is assumed to be a proper, convex, lower semicontinuous function whose subdifferential is denoted by ∂j. This problem is a model for real beam/shell applications.

MSC: 34B15; 49J40

Keywords: Fourth order equation; Nonsmooth nonlinearity; Critical point; Mountain pass theorem

1. Introduction

In this paper we investigate the following fourth order differential equation (inclusion):

$$u^{iv} - au'' + bu \in \overline{\partial} F(t, u), \tag{1}$$

$$\begin{pmatrix} -u'''(0) + au'(0) \\ u'''(1) - au'(1) \\ u''(0) \\ -u''(1) \end{pmatrix} \in \partial j \begin{pmatrix} u(0) \\ u(1) \\ u'(0) \\ u'(1) \end{pmatrix}, \tag{2}$$

where a, b are given real numbers, and functions F, j satisfy the following assumptions:

* Corresponding author. Fax: +36 1 3273166.
E-mail addresses: tphgyt01@phd.ceu.hu (T. Gyulov), morosanug@ceu.hu (G. Moroşanu).

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(H_1) $F(t, \xi) : (0, 1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, satisfying in addition $F(t, 0) = 0$ for a.a. $t \in (0, 1)$, as well as the Lipschitz condition:

$\forall \rho > 0$, there is an $\alpha_\rho \in L^1(0, 1)$ such that, for a.a. $t \in (0, 1)$ and all x, y with $|x|, |y| \le \rho$,

$$|F(t, x) - F(t, y)| \le \alpha_\rho(t)|x - y|;$$

(H_2) function $j : \mathbb{R}^4 \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous (l.s.c), such that $(0, 0, 0, 0)^T \in D(j)$.

Here $\overline{\partial} F(t, \xi)$ denotes the generalized Clarke gradient of $F(t, \cdot)$ at $\xi \in \mathbb{R}$, while ∂j stands for the subdifferential of j.

It is worth noting that our conditions $F(t, 0) = 0$ and $(0, 0, 0, 0)^T \in D(j)$ do not restrict the generality of the problem. It suffices to assume that $F(t, 0)$ is an L^1 function.

Problem (1) and (2) is motivated by the study of the following equations arising in the theory of elastic stability:

$$u^{iv}(t) = q, \quad t \in (0, 1), \tag{3}$$

$$Dw^{iv} + N_x w'' + Eh\frac{w}{a^2} = q, \quad t \in (0, 1). \tag{4}$$

Eq. (3) describes the deflection u of an elastic beam, while Eq. (4) models the radial deflection w for symmetrical buckling of a cylindrical shell under uniform axial compression N_x (see [14], p. 457, [8,12]). Here q describes the reaction of the support, which may depend nonlinearly on the deflection (see [2–6], [11,15]),

$$q(t) = f(t, u(t)),$$

or, more generally,

$$q(t) \in \overline{\partial} F(t, u(t)),$$

for some function F (when the nonlinearity in Eq. (1) has some jumps, e.g., the case of adhesive support; see [12]).

Condition (2) provides a general framework for different types of boundary conditions (see [7]). For example, the functional

$$j((x_1, x_2, x_3, x_4)^T) := \begin{cases} 0, & x_1 = x_2, x_3 = x_4, \\ +\infty, & \text{otherwise,} \end{cases}$$

corresponds to periodic conditions $u^{(i)}(0) = u^{(i)}(1), i = 0, 1, 2, 3$, while the case of simply supported endpoints, i.e., $u(0) = u(1) = u''(0) = u''(1) = 0$, can be obtained by choosing

$$j((x_1, x_2, x_3, x_4)^T) := \begin{cases} 0, & x_1 = x_2 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

In this paper we investigate the existence of solutions to problem (1) and (2). By a *solution* of this problem we mean a function $u \in C^3([0, 1], \mathbb{R})$ with absolutely continuous third derivative, which satisfies (2) and

$$u^{iv}(t) - au''(t) + bu(t) \in \overline{\partial} F(t, u) \quad \text{for a.a. } t \in (0, 1). \tag{5}$$

Consider the set

$$\mathcal{D} = \{u : u \in H^2(0, 1), (u(0), u(1), u'(0), u'(1))^T \in D(j)\},$$

and the functional $J : H^2(0, 1) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$J(u) := j((u(0), u(1), u'(0), u'(1))^T), \quad \forall u \in H^2(0, 1),$$

whose effective domain is $D(J) = \mathcal{D}$.

Obviously, $\mathcal{D} \neq \emptyset$ since $(0, 0, 0, 0)^T \in D(j)$, so J is proper, convex and l.s.c.

The following two constants,

$$\lambda_1 := \inf\left\{ \frac{\|u''\|^2_{L^2} + a\|u'\|^2_{L^2} + b\|u\|^2_{L^2}}{\|u\|^2_{L^2}} : u \in \mathcal{D} \setminus \{0\} \right\}, \tag{6}$$

and

$$\overline{\lambda}_1 := \lim_{s\to\infty} \inf_{\substack{tu\in\mathcal{D}\\ t\geq s}} \left\{ \|u''\|^2_{L^2} + a\,\|u'\|^2_{L^2} + b\|u\|^2_{L^2} + \frac{2J(tu)}{t^2} : \|u\|^2_{L^2} = 1 \right\}, \tag{7}$$

will be very important in the sequel.

It is easily seen that $\lambda_1 \leq \overline{\lambda}_1$, but in general $\lambda_1 < \overline{\lambda}_1$, e.g., if $a = b = 0$ and

$$j((x_1, x_2, x_3, x_4)^T) = \begin{cases} x_1^2, & x_1 = x_2, \\ +\infty, & \text{otherwise,} \end{cases}$$

then $\lambda_1 = 0$, and $\overline{\lambda}_1 > 0$. In this case, condition (2) corresponds to $u(0) = u(1), u''(0) = u''(1) = 0$, and $u'''(0) - au'(0) = u'''(1) - au'(1)$.

We are going to apply the variational approach developed in Motreanu and Panagiotopoulos [12] to the functional

$$I(u) := \frac{1}{2}\int_0^1 (u''^2 + au'^2 + bu^2)\mathrm{d}t - \int_0^1 F(t,u)\mathrm{d}t + J(u),$$

to obtain the existence of solutions to problem (1) and (2). Our present results generalize previous results in [5] in two directions. First, they involve nonsmooth nonlinearities in Eq. (1), which have definite physical meaning. Secondly, more general nonlinear boundary conditions are considered, which include many classical boundary conditions.

Our main results are included in the following two theorems.

Theorem 1. *Assume* (H1) *and* (H2). *If, in addition,*

(L_∞)

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{x^2} < \frac{\lambda_1}{2}, \tag{8}$$

uniformly for a.a. $t \in (0,1)$*, then problem* (1) *and* (2) *has at least a solution.*

Next, suppose that function F satisfies

(L_0)

$$\limsup_{x\to 0} \frac{F(t,x)}{x^2} < \frac{\lambda_1}{2},$$

uniformly for a.e. $t \in (0,1)$. Then, $0 \in \overline{\partial} F(t,0)$ for a.a. $t \in (0,1)$, so in this case $u(t) \equiv 0$ is a solution of problem (1) and (2). Thus it is reasonable to investigate the existence of nonzero solutions of problem (1) and (2). We have

Theorem 2. *Assume that* $\lambda_1 > 0$ *and that* (L_0), (H_1), *and* (H_2) *are fulfilled. Suppose, in addition, that* $D(j)$ *is closed,* $(0,0,0,0)^T \in \partial j((0,0,0,0)^T)$*, and either* (G_θ) *or* $(G_2) - (\overline{L}_\infty)$ *holds, where*

(G_θ) *there exist constants* $\theta > 2$*, and* $k, M > 0$ *such that*

$$j'(z;z) \leq \theta j(z) + k, \quad \forall z \in D(j), \tag{9}$$

$$0 < \theta F(t,x) \leq \xi x, \quad \forall \xi \in \overline{\partial} F(t,x), \tag{10}$$

for all $|x| > M$*, and a.a.* $t \in (0,1)$*,*

(G_2) *there exist positive constants* c, k, M *such that*

$$j'(z;z) \leq 2j(z) + k, \quad \forall z \in D(j), \tag{11}$$

$$0 < \left(2 + \frac{c}{|x|}\right) F(t,x) \leq \xi x, \quad \forall \xi \in \overline{\partial} F(t,x), \tag{12}$$

for all $|x| > M$*, and a.a.* $t \in (0,1)$*, and*

$(\overline{L}_\infty)$

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{x^2} > \frac{\overline{\lambda}_1}{2}, \tag{13}$$

uniformly for a.e. $t \in (0,1)$.

Then problem (1) and (2) has a nonzero solution.

Remark 3. Observe that any of the conditions (9) and (11) implies that the domain $D(j)$ of functional j is a convex cone. Moreover, assumption (11) guarantees that $\overline{\lambda}_1 < \infty$ (see Lemma 11).

Remark 4. Assumption (12) allows asymptotically quadratic growth of function $F(t,x)$ and implies that functional $I(\cdot)$ satisfies the Palais–Smale condition (Lemma 12). We may think about a more general condition of Cerami type, but it seems it is difficult to apply such a condition to the case when the functional is the sum of a locally Lipschitz functional and a proper, convex, l.s.c. one.

2. Preliminaries

Let X be a Banach space and let Φ be a locally Lipschitz function defined on X. Denote by $\Phi^0(u;v)$ the generalized directional derivative of Φ at point $u \in X$ in the direction $v \in X$,

$$\Phi^0(u;v) := \limsup_{w\to u, s\downarrow 0} \frac{\Phi(w+sv)-\Phi(w)}{s},$$

and by $\overline{\partial}\Phi(u)$ the generalized gradient of Clarke,

$$\overline{\partial}\Phi(u) = \{\eta \in X^* : \Phi^0(u;v) \ge \langle \eta, v\rangle, \forall v \in X\}.$$

Let us recall the definition of a critical point as well as the Palais–Smale condition for a functional I of the form

$$I = \Phi + \psi,$$

where Φ is as above, and $\psi : X \to (-\infty, +\infty]$ is a proper, convex and l.s.c. function.

Definition 5. An element $u \in X$ is called a critical point of the functional I if the following inequality holds:

$$\Phi^0(u; v-u) + \psi(v) - \psi(u) \ge 0, \quad \forall v \in X.$$

A number $c \in \mathbb{R}$ such that $I^{-1}(c)$ contains a critical point is called a critical value of I.

Definition 6. The functional I is said to satisfy the Palais–Smale condition if every sequence $\{u_n\} \subset X$ for which $I(u_n)$ is bounded and

$$\Phi^0(u_n; v-u_n) + \psi(v) - \psi(u_n) \ge -\varepsilon_n \|v-u_n\|, \quad \forall v \in X \tag{14}$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$, possesses a convergent subsequence.

We will use the following generalized mountain pass result which can be found in [12] (see also [10]).

Theorem 7 (*Mountain Pass*). *Suppose that I satisfies the (PS) condition, $I(0)=0$ and*

(i) *there exist $\alpha, \rho > 0$ such that $I(u) \ge \alpha$ if $\|u\| = \rho$,*

(ii) *$I(e) \le 0$ for some $e \in X$, with $\|e\| > \rho$.*

Then the number

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\},$$

is a critical value of I with $c \ge \alpha$.

In what follows, our space will be $X = H^2(0,1)$.
Define

$$\psi(u) = \frac{1}{2}\int_0^1 (u''^2 + u'^2 + u^2)\mathrm{d}t + J(u) = \frac{1}{2}\|u\|^2_{H^2(0,1)} + J(u), \quad u \in H^2(0,1),$$

whose effective domain is $D(\psi) = \mathcal{D}$, and

$$\Phi(u) := -\int_0^1 F(t,u)\mathrm{d}t + \varphi(u), \quad u \in H^2(0,1), \tag{15}$$

where

$$\varphi(u) := \frac{1}{2}\int_0^1 ((a-1)u'^2 + (b-1)u^2)\mathrm{d}t.$$

Obviously, ψ is proper, convex and l.s.c., while $\varphi \in C^1\left(H^2(0,1), \mathbb{R}\right)$, and

$$\langle \varphi'(u), v\rangle = \int_0^1 ((a-1)u'v' + (b-1)uv)\mathrm{d}t.$$

We have

Proposition 8. *Assume that $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfies* (H_1) *and let Φ be defined by* (15)*. Then $\Phi(\cdot)$ is locally Lipschitz. Moreover, if $u \in H^2(0,1)$ and $l \in \overline{\partial}\Phi(u)$ then there is some $u_l \in L^1(0,1)$ such that $u_l(t) \in \overline{\partial}F(t,u(t))$ for a.a. $t \in (0,1)$, and*

$$\langle l, v\rangle = \int_0^1 \left(-u_l v + (a-1)u'v' + (b-1)uv\right)\mathrm{d}t, \quad \forall v \in H^2(0,1), \tag{16}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing in $H^2(0,1)$.

Proof. The continuity of the embedding $H^2(0,1) \subset C^1([0,1])$ and assumption (H_1) imply that Φ is locally Lipschitz.
Define

$$R(u) := -\int_0^1 F(t,u(t))\mathrm{d}t, \quad \forall u \in H^2(0,1).$$

One may prove (see Theorem 2.7.3 in [1]) that given $\xi \in \overline{\partial}R(u)$ then there exists some $v_\xi \in L^1(0,1)$ such that $v_\xi(t) \in -\overline{\partial}F(t,u(t))$ for a.a. $t \in (0,1)$, and

$$\langle \xi, v\rangle = \int_0^1 v_\xi v\mathrm{d}t, \quad \forall v \in H^2(0,1). \tag{17}$$

For the reader's convenience we give a complete direct proof. First, Fatou's lemma and (H_1) imply that

$$R^0(u;v) \le \int_0^1 (-F(t,\cdot))^0(u(t);v(t))\mathrm{d}t \le \int_0^1 \alpha_\rho(t)|v(t)|\mathrm{d}t, \quad \forall v \in H^2(0,1), \tag{18}$$

where ρ is such that $\|u\|_C < \rho$, and $\alpha_\rho(t) \in L^1(0,1)$ is defined in (H_1).
Let $\xi \in \overline{\partial}R(u)$. Then

$$\langle \xi, v\rangle \le R^0(u;v) \le \int_0^1 \alpha_\rho(t)\,|v(t)|\,\mathrm{d}t, \quad \forall v \in H^2(0,1),$$

and the Hahn–Banach theorem implies that ξ can be extended to a continuous linear functional $\overline{\xi}$ on $L^1((0,1);\mu) \supset H^2(0,1)$, where μ is the measure on $[0,1]$, defined by

$$\mu(A) := \int_A \alpha_\rho(t)\mathrm{d}t,$$

for any set A, which is measurable with respect to the usual Lebesgue measure $m(\cdot)$. Obviously, $\alpha_\rho(t)$ can be chosen such that $\alpha_\rho(t) > 1$. So, $\mu(A) = 0$ iff $m(A) = 0$. Then, there exists $\beta_{\overline{\xi}} \in L^\infty((0,1);\mu) \equiv L^\infty(0,1)$ such that

$$\langle \overline{\xi}, w\rangle = \int_0^1 \alpha_\rho(t)\beta_{\overline{\xi}}(t)w(t)\mathrm{d}t, \quad \forall w \in L^1((0,1);\mu),$$

so

$$\langle \xi, v\rangle = \langle \overline{\xi}, v\rangle = \int_0^1 \alpha_\rho\,(t)\,\beta_{\overline{\xi}}(t)v(t)\mathrm{d}t, \quad \forall v \in H^2(0,1),$$

and (17) holds with $v_\xi := \alpha_\rho\beta_{\overline{\xi}} \in L^1(0,1)$. Now, (17) and (18) imply

$$\int_0^1 v_\xi v\mathrm{d}t \le \int_0^1 (-F(t,\cdot))^0\,(u(t); v(t))\,\mathrm{d}t, \quad \forall v \in H^2(0,1),$$

yielding that $v_\xi(t) \in \overline{\partial}(-F)(t,u(t)) = -\overline{\partial}F(t,u(t))$ for a.a. $t \in (0,1)$.

Finally, let $l \in \overline{\partial}\Phi(u)$. By $\overline{\partial}\Phi(u) \subset \overline{\partial}R(u)+\overline{\partial}\varphi(u) = \overline{\partial}R(u)+\varphi'(u)$, there exists $\xi \in \overline{\partial}R(u)$ such that $l = \xi+\varphi'(u)$ and (16) is obtained with $u_l := -v_\xi$, where v_ξ is determined by ξ as above. ■

Now, let the functional I be defined as follows

$$\begin{aligned} I(u) &= \Phi(u) + \psi(u) \\ &= \frac{1}{2}\int_0^1 (u''^2 + au'^2 + bu^2)\mathrm{d}t - \int_0^1 F(t,u)\mathrm{d}t + J(u). \end{aligned}$$

Theorem 9. *Assume that $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfies* (H_1) *and let $u \in H^2(0,1)$. If u is a critical point of functional I, then u is a solution of problem* (1) *and* (2).

Proof. We will adapt a previous device from [9], Proposition 3.2, to the present problem. If we take $v = u + sw$, $s > 0$, in the inequality

$$\Phi^0(u; v-u) + \psi(v) - \psi(u) \ge 0, \quad \forall v \in H^2(0,1),$$

and then divide by s and let $s \to 0^+$, we get

$$\Phi^0(u; w) + \int_0^1 \big(u''w'' + u'w' + uw\big)\,\mathrm{d}t + J'(u; w) \ge 0, \quad \forall w \in H^2(0,1), \tag{19}$$

where

$$J'(u; w) = j'((u(0), u(1), u'(0), u'(1))^T; (w(0), w(1), w'(0), w'(1))^T) \tag{20}$$

is the directional derivative of the convex function J at u in the direction w. If $w \in C_0^\infty(0,1) \subset H^2(0,1)$, inequality (19) becomes

$$\Phi^0(u; w) \ge -\int_0^1 (u''w'' + u'w' + uw)\mathrm{d}t, \quad \forall w \in C_0^\infty(0,1). \tag{21}$$

Function $\Phi^0(u; w\cdot)$ is subadditive and positively homogeneous, so by the Hahn–Banach theorem there exists a linear functional $l : H^2(0,1) \to \mathbb{R}$ such that

$$l(w) = -\int_0^1 \big(u''w'' + u'w' + uw\big)\,\mathrm{d}t, \quad \forall w \in C_0^\infty(0,1)$$

and

$$\Phi^0(u; w) \ge l(w), \quad \forall w \in H^2(0,1). \tag{22}$$

On the other hand, there exits a constant $k > 0$ such that

$$\Phi^0(u; w) \le k\|w\|_{H^2(0,1)}, \quad \forall w \in H^2(0,1), \tag{23}$$

which, together with (22), yields

$$|l(w)| \le k \, \|w\|_{H^2(0,1)}, \quad \forall w \in H^2(0,1),$$

showing that l is continuous and $l(w) = \langle l, w\rangle$. Now, from inequality (22), it follows that $l \in \overline{\partial}\Phi(u)$. Using Proposition 8, we deduce that there is some $u_l \in L^1(0,1)$ such that

$$u_l(t) \in \overline{\partial} F(t, u(t)), \quad \text{for a.a. } t \in (0,1), \tag{24}$$

and

$$\int_0^1 \left(u''w'' + au'w' + buw - u_l w\right) \mathrm{d}t = 0, \quad \forall w \in C_0^\infty(0,1). \tag{25}$$

As $u \in H^2(0,1)$, we have $u''' \in W^{1,1}(0,1)$, i.e. u''' is absolutely continuous and

$$u^{iv}(t) - au''(t) + bu(t) = u_l(t) \quad \text{for a.a. } t \in (0,1). \tag{26}$$

Now, from (24) it follows that (5) holds.

Next, we prove that u satisfies (2). The above inclusion (24) implies that

$$u_l(t)w(t) \le F^0(t, u(t); w(t)) \quad \text{for a.a. } t \in (0,1), \forall w \in H^2(0,1).$$

Then, by (26), we have

$$\begin{aligned}\int_0^1 &\left(u''w'' + au'w' + buw\right) \mathrm{d}t + \left(u'''(1) - au'(1)\right) w(1)\\ &- \left(u'''(0) - au'(0)\right) w(0) - u''(1)w'(1) + u''(0)w'(0)\\ &\le \int_0^1 F^0(t, u(t); w(t)) \mathrm{d}t,\end{aligned}$$

for all $w \in H^2(0,1)$. Thus, taking into account inequality (18), we have

$$\Phi^0(u; w) \le \int_0^1 (-F)^0(t, u(t); w(t)) \mathrm{d}t + \langle \varphi'(u), w\rangle,$$

and from (19), we get

$$\begin{aligned}\int_0^1 &(-F)^0 (t, u(t); w(t)) \, \mathrm{d}t - \int_0^1 F^0 (t, u(t); w(t)) \, \mathrm{d}t + J'(u; w)\\ &\ge \left(u'''(1) - au'(1)\right) w(1) - \left(u'''(0) - au'(0)\right) w(0) - u''(1)w'(1) + u''(0)w'(0),\end{aligned} \tag{27}$$

for all $w \in H^2(0,1)$. Now, let $x, y, z, q \in \mathbb{R}$ be arbitrarily chosen and, for each $n \in \mathbb{N}$, let $w_n \in H^2(0,1)$ be defined by

$$w_n := \begin{cases} x\omega_0(nt) + \dfrac{y}{n}\omega_1(nt), & \text{if } t \in \left[0, \dfrac{1}{n}\right), \\ 0, & \text{if } t \in \left[\dfrac{1}{n}, 1 - \dfrac{1}{n}\right], \\ z\omega_0 \left(n\left(1-t\right)\right) - \dfrac{q}{n}\omega_1 \left(n\left(1-t\right)\right), & \text{if } t \in \left(1 - \dfrac{1}{n}, 1\right], \end{cases}$$

where $\omega_0(s)$ and $\omega_1(s)$ are such that $\omega_0(1) = \omega_1(1) = \omega_0'(1) = \omega_1'(1) = 0, \omega_0(0) = \omega_1'(0) = 1$ and $\omega_0'(0) = \omega_1(1) = 0$, e.g., $\omega_0(s) := (s-1)^2(2s+1)$ and $\omega_1(s) := s(s-1)^2$.

Then, $w_n(0) = x$, $w_n'(0) = y$, $w_n(1) = z$, and $w_n'(1) = q$. From hypothesis (H_1) there is some $\alpha_\rho \in L^1(0,1)$ such that

$$|F^0(t, u(t); \eta)| \le \alpha_\rho(t) \, |\eta| \, , \quad \forall \eta \in \mathbb{R}, \text{ for a.a. } t \in (0,1),$$

where $\rho > 0$ depends on the supremum norm $\|u\|_C$ of u. Taking $\eta = w_n(t)$, one obtains

$$|F^0(t, u(t); w_n(t))| \le \alpha_\rho(t) \max\left\{|x| + \frac{4|y|}{27n}, |z| + \frac{4|q|}{27n}\right\}, \quad \text{for a.a. } t \in (0, 1). \tag{28}$$

On the other hand,

$$F^0(t, u(t); w_n(t)) \to F^0(t, u(t); 0) = 0, \quad \text{for a.a. } t \in (0, 1).$$

This together with (28) implies, by Lebesgue's dominated convergence theorem, that

$$\int_0^1 F^0(t, u(t); w_n(t))\mathrm{d}t \to 0, \quad \text{as } n \to \infty. \tag{29}$$

Similarly, one has

$$\int_0^1 (-F)^0\,(t, u\,(t)\,;\, w_n(t))\,\mathrm{d}t \to 0, \quad \text{as } n \to \infty. \tag{30}$$

In (27) we take $w = w_n$ and let $n \to \infty$. Thus, by (20), (29) and (30), we derive

$$\begin{aligned} &j'((u(0), u(1), u'(0), u'(1))^T; (x, z, y, q)^T) \\ &\quad \ge (u'''(1) - au'(1))z - (u'''(0) - au'(0))x - u''(1)q + u''(0)y, \end{aligned}$$

which, as x, y, z, and q were arbitrarily chosen, implies that u satisfies (2) (see Theorem 23.2 in [13]). ■

Lemma 10. *Define λ_1 by (6). Then $\lambda_1 > -\infty$. Moreover, if $\lambda_1 > 0$, then there exists a constant $m > 0$ such that*

$$\|u''\|_{L^2}^2 + a\,\|u'\|_{L^2}^2 + b\|u\|_{L^2}^2 \ge m\|u\|_{H^2(0,1)}^2.$$

Proof. Indeed, there exists a constant K such that

$$\|u'\|_{L^2}^2 \le K\left(\varepsilon\,\|u''\|_{L^2}^2 + \varepsilon^{-1}\|u\|_{L^2}^2\right), \quad \forall \varepsilon \in (0, 1)\,. \tag{31}$$

Choose ε such that $K\varepsilon \le |a|^{-1}$ and λ great enough such that $b + \lambda \ge |a|\,K\varepsilon^{-1}$. Then,

$$\|u'\|_{L^2}^2 \le K\left(\varepsilon\,\|u''\|_{L^2}^2 + \varepsilon^{-1}\|u\|_{L^2}^2\right) \le |a|^{-1}\left(\|u''\|_{L^2}^2 + (b+\lambda)\,\|u\|_{L^2}^2\right),$$

i.e.,

$$\|u''\|_{L^2}^2 - |a|\,.\,\|u'\|_{L^2}^2 + b\|u\|_{L^2}^2 \ge -\lambda\|u\|_{L^2}^2,$$

so $\lambda_1 \ge -\lambda > -\infty$.

Next, suppose that $\lambda_1 > 0$ holds. We will prove that there is some $\mu > 0$ such that

$$\|u''\|_{L^2}^2 + a\,\|u'\|_{L^2}^2 + b\,\|u\|_{L^2}^2 \ge \frac{\mu}{\mu+1}\left(\|u''\|_{L^2}^2 + \|u'\|_{L^2}^2 + \|u\|_{L^2}^2\right), \tag{32}$$

which is equivalent to

$$\|u''\|_{L^2}^2 + ((a-1)\,\mu + a)\,\|u'\|_{L^2}^2 + ((b-1)\mu + b)\,\|u\|_{L^2}^2 \ge 0.$$

Suppose that $a \ne 1$. Take $\mu > 0$ small enough such that

$$\lambda_1 + (b-1)\mu + \frac{2K^2\,(a-1)^2\,\mu^2}{\lambda_1}\,(b - \lambda_1) \ge \frac{\lambda_1}{2}\left(1 + \frac{2K^2a(a-1)\mu}{\lambda_1}\right)^2,$$

and

$$1 > \delta := \frac{2K^2(a-1)^2\mu^2}{\lambda_1}, \qquad 1 > \varepsilon := \frac{2K\,|a-1|\,\mu}{\lambda_1 + 2K^2a(a-1)\mu} > 0,$$

where K is defined as in (31). Thus, inequality (32) is equivalent to

$$(1-\delta)\left(\left\|u''\right\|_{L^2}^2 + a\left\|u'\right\|_{L^2}^2 + b\|u\|_{L^2}^2\right) + \delta\left\|u''\right\|_{L^2}^2 + ((a-1)\mu+\delta a)\left\|u'\right\|_{L^2}^2 + ((b-1)\mu+\delta b)\|u\|_{L^2}^2 \geq 0.$$

Now, we have

$$(a-1)\mu+\delta a = (a-1)\mu + \frac{2K^2(a-1)^2\mu^2}{\lambda_1}a = \frac{(a-1)\mu}{\lambda_1}(\lambda_1 + 2K^2 a(a-1)\mu),$$
$$\delta - |(a-1)\mu+\delta a|\,K\varepsilon = 0,$$
$$|(a-1)\mu+\delta a|\,K\varepsilon^{-1} = \frac{\lambda_1}{2}\left(1+\frac{2K^2a(a-1)\mu}{\lambda_1}\right)^2,$$

and hence

$$\begin{aligned}(1-\delta)&\left(\left\|u''\right\|_{L^2}^2 + a\left\|u'\right\|_{L^2}^2 + b\|u\|_{L^2}^2\right) + \delta\left\|u''\right\|_{L^2}^2 + ((a-1)\mu+\delta a)\left\|u'\right\|_{L^2}^2 + ((b-1)\mu+\delta b)\|u\|_{L^2}^2 \\ &\geq (1-\delta)\lambda_1\|u\|_{L^2}^2 + \delta\left\|u''\right\|_{L^2}^2 - |(a-1)\mu+\delta a|\left\|u'\right\|_{L^2}^2 + ((b-1)\mu+\delta b)\|u\|_{L^2}^2 \\ &\geq (\lambda_1 + (b-1)\mu + \delta(b-\lambda_1))\|u\|_{L^2}^2 + \delta\left\|u''\right\|_{L^2}^2 - |(a-1)\mu+\delta a|\,K\left(\varepsilon\left\|u''\right\|_{L^2}^2 + \varepsilon^{-1}\|u\|_{L^2}^2\right) \geq 0.\end{aligned}$$

Finally, let $a = 1$. Choose $\mu > 0$ such that $(b-1)\mu > -\lambda_1$. Then (32) is equivalent to

$$\left\|u''\right\|_{L^2}^2 + \left\|u'\right\|_{L^2}^2 + ((b-1)\mu+b)\|u\|_{L^2}^2 \geq 0,$$

which is obviously true because

$$\left\|u''\right\|_{L^2}^2 + \left\|u'\right\|_{L^2}^2 + b\|u\|_{L^2}^2 \geq \lambda_1\|u\|_{L^2}^2. \quad \blacksquare$$

Lemma 11. *Assume that (11) holds. Then* $\overline{\lambda}_1 < +\infty$.

Proof. A standard computation shows that

$$s^{-2}j(sz) \leq j(z) + \frac{k}{2}(1-s^{-2}), \quad \forall z \in D(j),\ \forall s \geq 1,$$

which yields

$$\overline{\lambda}_1 \leq \inf\left\{\left\|u''\right\|_{L^2}^2 + a\left\|u'\right\|_{L^2}^2 + b\|u\|_{L^2}^2 + 2J(u) : u \in H^2(0,1)\ \|u\|_{L^2}^2 = 1, (u(0), u(1), u'(0), u'(1))^T \in D(j)\right\} + k. \quad \blacksquare$$

Proof of Theorem 1. First, we will prove that functional I is coercive. Suppose that, on the contrary, there exist a constant $C > 0$ and a sequence $\{u_n\} \subset H^2(0,1)$ such that $\|u_n\|_{H^2(0,1)} \to \infty$ and $I(u_n) \leq C$. Define $v_n := \frac{u_n}{\|u_n\|}$, where $\|\cdot\|$ is the norm in $H^2(0,1)$. Then, $u_n, v_n \in D(J)$. Moreover, $\|v_n\| = 1$ implies that there exists a subsequence of $\{v_n\}$ (denoted again by $\{v_n\}$) and $v \in D(J)$ such that $v_n \rightharpoonup v$ in $H^2(0,1)$. Therefore, $v_n \to v$ strongly in $C^1([0,1])$.

Since J is proper, convex and l.s.c., it is bounded from below by an affine functional. Therefore, there are some constants $c_1 > 0$ and $c_2 > 0$ such that

$$J(u) \geq -c_1 - c_2\|u\|. \tag{33}$$

So, we have

$$\begin{aligned}C &\geq \frac{1}{2}\int_0^1\left(u_n''^2 + au_n'^2 + bu_n^2\right)\mathrm{d}t + J(u_n) + \Phi(u_n) - \varphi(u_n) \\ &\geq \frac{1}{2}\int_0^1\left(u_n''^2 + au_n'^2 + bu_n^2\right)\mathrm{d}t + \Phi(u_n) - \varphi(u_n) - c_1 - c_2\|u_n\|,\end{aligned}$$

which implies

$$\frac{C+c_1}{\|u_n\|^2}+\frac{c_2}{\|u_n\|}\geq\frac{1}{2}\int_0^1\left(v_n''^2+av_n'^2+bv_n^2\right)\mathrm{d}t+\frac{\Phi(u_n)-\varphi(u_n)}{\|u_n\|^2}. \tag{34}$$

If (8) holds, then there are constants $\sigma>0$ and $\rho>0$ such that

$$F(t,x)\leq\frac{\lambda_1-\sigma}{2}x^2\quad\text{for a.a. } t\in(0,1),$$

and for all x with $|x|>\rho$. Hence,

$$F(t,x)\leq\rho\alpha_\rho(t)+\left|\frac{\lambda_1-\sigma}{2}\right|\rho^2+\frac{\lambda_1-\sigma}{2}x^2\quad\text{for a.a. } t\in(0,1),\, x\in\mathbb{R},$$

which gives

$$\Phi(u)-\varphi(u)=-\int_0^1F(t,u)\mathrm{d}t\geq-k-\frac{\lambda_1-\sigma}{2}\int_0^1u^2\mathrm{d}t,\quad\forall u\in H^2(0,1),$$

so by (34) it follows that

$$\frac{C+c_1+k}{\|u_n\|^2}+\frac{c_2}{\|u_n\|}\geq\frac{1}{2}\int_0^1\left(v_n''^2+av_n'^2+(b-\lambda_1+\sigma)\,v_n^2\right)\mathrm{d}t\geq\frac{\sigma}{2}\int_0^1v_n^2\mathrm{d}t. \tag{35}$$

Therefore, $v_n\to v=0$ in $C^1([0,1])$, and this, together with

$$\|v_n''\|_{L^2}^2+\|v_n'\|_{L^2}^2+\|v_n\|_{L^2}^2=\|v_n\|_{H^2}^2=1,$$

implies that

$$\begin{aligned}\frac{C+c_1+k}{\|u_n\|^2}+\frac{c_2}{\|u_n\|}\geq&\ \frac{1}{2}+\varphi(v_n)-\frac{\lambda_1-\sigma}{2}\int_0^1v_n^2\mathrm{d}t\\ &\to\frac{1}{2}+\varphi(v)-\frac{\lambda_1-\sigma}{2}\int_0^1v^2\mathrm{d}t=\frac{1}{2},\end{aligned}$$

a contradiction.

Finally, by the compact embedding $H^2(0,1)\subset C^1([0,1])$, functional Φ is sequentially weakly continuous, and ψ is weakly lower semicontinuous. Hence, I is sequentially weakly lower semicontinuous and its coercivity implies that it is bounded from below and attains its infimum. So I has a critical point, which by Theorem 9 is a solution of problem (1) and (2). ■

Lemma 12. *Assume* (H$_1$) *holds,* $\lambda_1>0$*, and either* (G_θ) *or* (G_2) *holds. If, in addition,* $D(j)$ *is closed, then functional* I *satisfies the Palais–Smale condition.*

Proof. First, we prove that each Palais–Smale sequence is bounded. Let $\{u_n\}$ be such a sequence. Then, there exists a constant C such that

$$C\geq\frac{1}{2}\int_0^1\left(u_n''^2+au_n'^2+bu_n^2\right)\mathrm{d}t+\Phi(u_n)-\varphi(u_n)+J(u_n). \tag{36}$$

On the other hand, setting $v=(1+s)u_n$, $s>0$, in (14) and taking the limit as $s\to0^+$, we obtain

$$\Phi^0(u_n;u_n)+\psi'(u_n;u_n)\geq-\varepsilon_n\|u_n\|.$$

This inequality reads

$$\Phi^0(u_n;u_n)-\langle\varphi'(u_n),u_n\rangle+\int_0^1\left(u_n''^2+au_n'^2+bu_n^2\right)\mathrm{d}t+J'(u_n;u_n)\geq-\varepsilon_n\|u_n\|. \tag{37}$$

We will examine separately the cases when (G_2) and (G_θ) hold.

Case 1. Let (G_θ) hold for some $\theta > 2$. We will prove that

$$\theta(\Phi(u) - \varphi(u)) \geq \Phi^0(u; u) - \langle \varphi'(u), u \rangle - m_1, \quad \forall u \in H^2(0,1), \tag{38}$$

and

$$\theta J(u) \geq J'(u; u) - m_2, \quad \forall u \in H^2(0,1), \tag{39}$$

for some positive constants m_1 and m_2. Indeed, (39) follows from (9) and the definition of functional J. Now, let $l \in \overline{\partial}\Phi(u)$. By Proposition 8, there exists some $u_l \in L^1$ such that $u_l(t) \in \overline{\partial} F(t, u(t))$ for a.a. $t \in (0,1)$, and (16) holds. Hypothesis (H_1) implies that given $M > 0$ there exists an $\alpha_M(t) \in L^1$ such that for each $x \in \mathbb{R}$, with $|x| \leq M$, the inequalities

$$|\xi| \leq \alpha_M(t), \quad \forall \xi \in \overline{\partial} F(t, x),$$

and

$$|F(t, x)| \leq M\alpha_M(t),$$

are satisfied. Hence,

$$\begin{aligned}\int_0^1 u(t)u_l(t)\mathrm{d}t &= \int_{|u(t)|>M} u(t)\, u_l(t)\mathrm{d}t + \int_{|u(t)|\leq M} u(t)u_l(t)\mathrm{d}t \\ &\geq \int_{|u(t)|>M} \theta F(t, u(t))\mathrm{d}t - M\int_0^1 \alpha_M(t)\mathrm{d}t \\ &= \theta\left(\int_0^1 F(t, u(t))\,\mathrm{d}t - \int_{|u(t)|\leq M} F(t, u(t))\,\mathrm{d}t\right) \\ -M\int_0^1 \alpha_M(t)\mathrm{d}t &\geq \theta \int_0^1 F(t, u(t))\,\mathrm{d}t - M(1+\theta)\int_0^1 \alpha_M(t)\mathrm{d}t,\end{aligned}$$

which yields

$$-\langle l, u\rangle + \langle \varphi'(u), u\rangle \geq \theta\left(-\Phi(u) + \varphi(u)\right) - M(1+\theta)\int_0^1 \alpha_M(t)\mathrm{d}t,$$

i.e.,

$$\theta(\Phi(u) - \varphi(u)) \geq \langle l, u\rangle - \langle \varphi'(u), u\rangle - m_1, \quad \forall l \in \overline{\partial}\Phi(u),$$

where $m_1 = M(1+\theta)\int_0^1 \alpha_M(t)\mathrm{d}t > 0$. It follows that

$$\begin{aligned}\theta(\Phi(u) - \varphi(u)) &\geq \max\left\{\langle l, v\rangle : l \in \overline{\partial}\Phi(u)\right\} - \langle \varphi'(u), u\rangle - m_1 \\ &= \Phi^0(u; u) - \langle \varphi'(u), u\rangle - m_1,\end{aligned}$$

which yields (38).

Now, multiplying (36) by $-\theta$ and adding (37) to (38) and (39), we find

$$\theta C + m_1 + m_2 \geq \left(\frac{\theta}{2} - 1\right)\int_0^1 \left(u_n''^2 + a u_n'^2 + b u_n^2\right)\mathrm{d}t - \varepsilon_n \|u_n\|.$$

On the other hand, by the hypothesis $\lambda_1 > 0$ and by Lemma 10, there exists a constant $m_3 > 0$ such that

$$\theta C + m_1 + m_2 \geq m_3\|u_n\|^2 - \varepsilon_n\|u_n\|,$$

which implies that $\{u_n\}$ is bounded.

Case 2. Let (G_2) hold. We will prove that there exists a constant $k_1 > 0$, and, given $\rho > 0$ there exists a constant $m_1 = m_1(\rho) > 0$, such that for each $u \in H^2(0,1)$, with $\|u\| \geq \rho$, the inequality

$$\left(2 + \frac{k_1}{\|u\|}\right)(\Phi(u) - \varphi(u)) \geq \Phi^0(u; u) - \langle \varphi'(u), u\rangle - m_1 \tag{40}$$

holds. Let $l \in \overline{\partial}\Phi(u)$ and $u_l \in L^1$ be defined as in Proposition 8. Then, we have

$$\begin{aligned}\int_0^1 u(t)u_l(t)\mathrm{d}t &= \int_{\{|u(t)|>M\}} u(t)u_l(t)\mathrm{d}t + \int_{\{|u(t)|\le M\}} u(t)u_l(t)\mathrm{d}t \\ &\ge \int_{\{|u(t)|>M\}} \left(2+\frac{c}{|u(t)|}\right) F(t,u(t)) - M\int_0^1 \alpha_M(t)\mathrm{d}t \\ &\ge \left(2+\frac{c}{d\|u\|}\right)\int_{\{|u(t)|>M\}} F(t,u(t)) - M\int_0^1 \alpha_M(t)\mathrm{d}t \\ &= \left(2+\frac{c}{d\|u\|}\right)\left(\int_0^1 F(t,u(t)) - \int_{\{|u(t)|\le M\}} F(t,u(t))\right) \\ -M\int_0^1 \alpha_M(t)\mathrm{d}t &\ge \left(2+\frac{c}{d\|u\|}\right)\int_0^1 F(t,u(t)) - M\left(3+\frac{c}{d\rho}\right)\int_0^1 \alpha_M(t)\mathrm{d}t,\end{aligned}$$

where we have used the obvious inequality

$$|u(t)| \le d\,\|u\|_{H^2(0,1)}$$

for some positive constant d. Similarly to in Case 1, we obtain that (40) holds with $k_1 = c/d$, and

$$m_1 = M\left(3+\frac{c}{\mathrm{d}\rho}\right)\int_0^1 \alpha_M(t)\mathrm{d}t.$$

Now, suppose that $\{u_n\}$ is unbounded. We may assume that $\|u_n\| \ge 1$ for all n. Then, applying (40) with $\rho = 1$ to (36) and (37), we get

$$2C + \frac{\tilde{C}}{\|u_n\|} \ge \frac{k_1}{2\|u_n\|}\int_0^1 \left(u_n''^2 + au_n'^2 + bu_n^2\right)\mathrm{d}t - \overline{m}_1 - \varepsilon_n\|u_n\|, \tag{41}$$

for some constants $\tilde{C} > 0$ and $\overline{m}_1 > 0$. Here we have used (11) and (33), i.e.,

$$\begin{aligned}\left(2+\frac{k_1}{\|u\|}\right)J(u) &\ge J'(u;u) - k + \frac{k_1}{\|u\|}J(u) \\ &\ge J'(u;u) - k - k_1\left(c_2 + \frac{c_1}{\|u\|}\right), \quad \forall u \in H^2(0,1),\ \|u\| \ge 1.\end{aligned}$$

Now, by $\lambda_1 > 0$ and Lemma 10, inequality (41) implies

$$2C + \tilde{C} + \overline{m}_1 \ge \left(\frac{k_1 m}{2} - \varepsilon_n\right)\|u_n\|,$$

a contradiction since $\varepsilon_n \to 0$ and $\|u_n\| \to \infty$. Therefore $\{u_n\}$ is bounded.

Next, let $\{u_n\}$ again be a Palais–Smale sequence. By the boundedness of $\{u_n\}$, under each of the hypotheses (G_θ) and (G_2), there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, and $u \in H^2(0,1)$, such that $u_n \rightharpoonup u$ in $H^2(0,1)$. Thus, $u_n \to u$ strongly in $C^1([0,1])$ and, as $D(j)$ is convex and closed, $u \in D(J)$. From (14) we derive that

$$\Phi^0(u_n; u-u_n) + J'(u_n; u-u_n) + \varepsilon_n\|u-u_n\| + \langle u_n, u\rangle \ge \|u_n\|^2.$$

Obviously, (15) allows Φ to be extended to a locally Lipschitz functional $\overline{\Phi}$ defined on $C^1([0,1])$. Moreover, by

$$\Phi^0(v;w) = \overline{\Phi}^0(v;w), \quad \forall v, w \in H^2(0,1),$$

and the upper semicontinuity of $\overline{\Phi}^0(\cdot;\cdot)$, it follows that

$$\limsup_{n\to\infty} \Phi^0(u_n; u-u_n) \le \overline{\Phi}^0(u;0) = 0.$$

On the other hand,

$$\limsup_{n\to\infty} J'(u_n; u-u_n) \le \limsup_{n\to\infty} (J(u) - J(u_n)) = J(u) - \liminf_{n\to\infty} J(u_n) \le 0.$$

Hence

$$\|u\|^2 \ge \limsup_{n\to\infty}(\Phi^0(u_n; u-u_n) + J'(u_n; u-u_n) + \varepsilon_n \|u-u_n\| + \langle u_n, u\rangle) \ge \liminf_{n\to\infty} \|u_n\|^2 \ge \|u\|^2,$$

i.e., $\|u_n\| \to \|u\|$. So, $u_n \to u$ strongly in $H^2(0,1)$. Thus, I satisfies the Palais–Smale condition, as claimed. ■

Proof of Theorem 2. We will apply Theorem 7. First of all, according to Lemma 12, I satisfies the Palais–Smale condition.

Next, we can assume without any loss of generality that $j((0,0,0,0)^T) = 0$. Since $(0,0,0,0)^T \in \partial j((0,0,0,0)^T)$, we have $J(u) \ge J(0) = 0$ and so in particular $I(0) = 0$. Now, we will prove that there exist $\rho > 0$ and $\alpha(\rho) > 0$ such that $I(u) \ge \alpha$ for $\forall u \in H^2(0,1)$ with $\|u\| = \rho$, where $\|\cdot\|$ denotes the norm of $H^2(0,1)$. Indeed, if $\rho = \|u\|$ is small enough, then by $|u(t)| \le d\|u\|$ for some constant d, and by

$$F(t,x) \le \frac{\lambda_1 - \sigma}{2} x^2, \quad \forall |x| \le \delta,$$

for some constants $\sigma > 0$ and $\delta > 0$, we have

$$I(u) = \frac{1}{2}\int_0^1 (u''^2 + au'^2 + bu^2)\mathrm{d}t - \int_0^1 F(t,u)\mathrm{d}t + J(u) \ge \frac{1}{2}\int_0^1 (u''^2 + au'^2 + bu^2)\mathrm{d}t - \frac{\lambda_1 - \sigma}{2}\int_0^1 u^2 \mathrm{d}t \ge m\|u\|^2,$$

for some constant $m > 0$. Here, we have used the inequality

$$\int_0^1 (u''^2 + au'^2 + (b - \lambda_1 + \sigma)u^2)\mathrm{d}t \ge \sigma \int_0^1 u^2 \mathrm{d}t,$$

as well as Lemma 10 with b replaced by $b - \lambda_1 + \sigma$.

Finally, we have to find an $e \in H^2(0,1)$ such that

$$I(e) \le 0 \tag{42}$$

and

$$\|e\| > \rho. \tag{43}$$

In what follows, we will examine separately the two alternative cases of the theorem, which we denote by (a) and (b).

(a) (G_θ) holds.

The mapping $s \longmapsto s^{-\theta}F(t,sx)$ is locally Lipschitz for a.a. $t \in (0,1)$, so we have for each $s > 0$

$$\overline{\partial}_s(s^{-\theta}F(t,sx)) \subset \overline{\partial}_s(s^{-\theta}F(t,sx)) + s^{-\theta}\overline{\partial}_s(F(t,sx)) = s^{-\theta-1}(-\theta F(t,sx) + sx\overline{\partial}F(t,sx)).$$

Let $|x| > M$, where M is the constant which appears in the statement of the theorem. Given $1 \le r < s$, by Lebourg's mean value theorem and assumption (10), there exist $\tau \in (r,s)$ and $\xi \in \overline{\partial}_s(s^{-\theta}F(t,sx))|_{s=\tau}$, $\xi \ge 0$ such that

$$s^{-\theta}F(t,sx) - r^{-\theta}F(t,rx) = \xi(s-r) \ge 0,$$

i.e.,

$$F(t,sx) \ge s^\theta F(t,x), \quad \text{for a.a. } t \in [0,1],\ \forall |x| > M,\ s \ge 1.$$

Now, let $h \in C_0^\infty(0,1)$ be such that $|h| > M$ on a set with positive measure. We have $J(sh) = 0$ for each s, and hence

$$\begin{aligned} I(sh) &= \frac{s^2}{2}\int_0^1 (h''^2 + ah'^2 + bh^2)\mathrm{d}t - \int_0^1 F(t, sh)\mathrm{d}t \\ &= \frac{s^2}{2}\int_0^1 (h''^2 + ah'^2 + bh^2)\mathrm{d}t - \int_{\{|sh|>M\}} F(t, sh)\mathrm{d}t - \int_{\{|sh|\le M\}} F(t, sh)\mathrm{d}t \\ &\le \frac{s^2}{2}\int_0^1 (h''^2 + ah'^2 + bh^2)\mathrm{d}t - \int_{\{|h|>M\}} F(t, sh)\,\mathrm{d}t + M\int_0^1 \alpha_M(t)\mathrm{d}t \\ &\le \frac{s^2}{2}\int_0^1 (h''^2 + ah'^2 + bh^2)\mathrm{d}t - s^\theta \int_{\{|h|>M\}} F(t, h)\mathrm{d}t + M\int_0^1 \alpha_M(t)\mathrm{d}t, \end{aligned}$$

for all $s \ge 1$, i.e.,

$$I(sh) \le s^2 k_1 - s^\theta k_2 + k_3 \to -\infty, \quad \text{as } s \to \infty,$$

with $k_1, k_2, k_3 > 0$. Therefore, we can choose s_0 sufficiently large such that $I(s_0 h) \le 0$ and $\|s_0 h\| > \rho$. Then $e := s_0 h$ satisfies conditions (42) and (43).

(b): (G_2) and $(\overline{L}_\infty)$ hold.

Let $u_n \in \mathcal{D}$ and $s_n > 0$ be such that $\|u_n\|_{L^2}^2 = 1$, $s_n \to \infty$,

$$\|u_n''\|_{L^2}^2 + a\,\|u_n'\|_{L^2}^2 + b\,\|u_n\|_{L^2}^2 + \frac{2J(s_n u_n)}{s_n^2} \to \overline{\lambda}_1.$$

On the other hand, by (13) there exist constants $C > 0$ and $\sigma > 0$ such that

$$F(t,x) \ge \frac{\overline{\lambda}_1 + \sigma}{2} x^2, \quad \forall |x| > C, \text{ a.a. } t \in (0,1).$$

We have

$$\begin{aligned} \int_0^1 F(t, s_n u_n(t))\,\mathrm{d}t &= \int_{\{|s_n u_n|>C\}} F(t, s_n u_n)\,\mathrm{d}t + \int_{\{|s_n u_n|\le C\}} F(t, s_n u_n)\,\mathrm{d}t \\ &\ge s_n^2 \frac{\overline{\lambda}_1 + \sigma}{2} \int_{\{|s_n u_n|>C\}} u_n^2 \mathrm{d}t - C\int_0^1 \alpha_C(t)\mathrm{d}t \\ &= s_n^2 \frac{\overline{\lambda}_1 + \sigma}{2} \left(\int_0^1 u_n^2 \mathrm{d}t - \int_{\{|s_n u_n|\le C\}} u_n^2 \mathrm{d}t \right) - C\int_0^1 \alpha_C(t)\mathrm{d}t \\ &\ge s_n^2 \frac{\overline{\lambda}_1 + \sigma}{2} \int_0^1 u_n^2 \mathrm{d}t - \left| \frac{\overline{\lambda}_1 + \sigma}{2} \right| C^2 - C\int_0^1 \alpha_C(t)\mathrm{d}t, \end{aligned}$$

and hence

$$I(s_n u_n) \le \frac{s_n^2}{2}\int_0^1 \left(u_n''^2 + a u_n'^2 + \left(b - \overline{\lambda}_1 - \sigma\right) u_n^2 \right) \mathrm{d}t + J(s_n u_n) - \left| \frac{\overline{\lambda}_1 + \sigma}{2} \right| C^2 - C\int_0^1 \alpha_C(t)\mathrm{d}t,$$

i.e.,

$$\frac{I(s_n u_n)}{s_n^2} \le \frac{1}{2}\left(\int_0^1 \left(u_n''^2 + a u_n'^2 + b u_n^2 \right)\mathrm{d}t + 2\frac{J(s_n u_n)}{s_n^2} \right) - \frac{\left(\overline{\lambda}_1 + \sigma\right)}{2} \int_0^1 u_n^2 \mathrm{d}t - \frac{C_1}{s_n^2},$$

which converges to $-\sigma/2$ as $n \to \infty$. Therefore, one can choose some $n = m$ such that $I(s_m u_m) < 0$ and $\|s_m u_m\| > \rho$. This is possible, since $\|u_n\|_{L^2(0,1)} = 1$, and $s_n \to \infty$. Obviously, $e := s_m u_m$ satisfies (42) and (43). ■

3. An example

Consider the boundary value problem

$$u^{iv} = F'(u), \tag{44}$$
$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{45}$$

where $F(x) = \frac{c}{2}x^2 e^{-\frac{1}{|x|}}$, $c > 0$. It is easily seen that

$$F'(x) = c\left(x + \frac{\operatorname{sign} x}{2}\right) e^{-\frac{1}{|x|}}$$

and we can choose $j((x_1, x_2, x_3, x_4)^T) = 0$, if $x_1 = x_2 = x_3 = x_4 = 0$, and $= +\infty$, otherwise.

Obviously, functions j and F satisfy assumptions (11), (12), and $\lim_{|x|\to\infty} \frac{F(x)}{x^2} = c/2$. Note that $\lambda_1 = \overline{\lambda}_1 > 0$. In fact, λ_1 is the first eigenvalue of the clamped beam operator.

If $c \le \lambda_1$, then $\forall u \in D(J)$,

$$I(u) = \frac{1}{2}\left(\int_0^1 u''^2 \mathrm{d}t - c\int_0^1 u^2 e^{-\frac{1}{|u|}} \mathrm{d}t\right),$$

and

$$\begin{aligned} 0 &= I'(u; u), \\ &= \int_0^1 \left(u''^2 - \lambda_1 u^2\right) \mathrm{d}t + \lambda_1 \int_0^1 u^2 \mathrm{d}t - c\int_0^1 \left(u^2 + \frac{|u|}{2}\right) e^{-\frac{1}{|u|}} \mathrm{d}t \\ &\ge c\int_0^1 u^2 \left(1 - \left(1 + \frac{1}{2|u|}\right) e^{-\frac{1}{|u|}}\right) \mathrm{d}t \ge 0, \end{aligned}$$

where $I'(u; u)$ is the directional derivative of I at u in the direction u. It follows that problem (44) and (45) has only the null solution.

If $c > \lambda_1$, then Theorem 2 guarantees the existence of at least one nonzero solution.

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