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On a singularly perturbed, coupled parabolic–parabolic problem

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Abstract. In this paper we study a model for convection–diffusion processes in which (1) the characters of both diffusion and convection change discontinuously at an internal domain point, (2) there is a small parameter ε, making it a singular perturbation problem, and (3) one of the boundary conditions is nonlinear. Specifically, the problem is (S_ε), (BC_ε), (IC_ε), (TC_ε) formulated below. The problem is singularly perturbed with respect to the uniform convergence topology, with an internal transition layer. An asymptotic expansion of order zero for the solution is determined formally. Then some estimates for the remainder components are established to validate our expansion.

Keywords: singularly perturbed, coupled, parabolic, convection–diffusion, asymptotic expansion, internal transition layer

1. Introduction

Let $D = (a, c) \times (0, T)$ be a given rectangle in the (x, t)-plane $(-\infty < a < c < +\infty,\ 0 < T < +\infty)$ and let $D_1 = (a, b) \times (0, T),\ D_2 = (b, c) \times (0, T)$ be two sub-rectangles of D $(a < b < c)$. We consider in D the following coupled parabolic–parabolic problem, which will be called P_ε:

$$\begin{cases} u_t - \varepsilon u_{xx} + \alpha(x)u_x + \beta(x)u = f(x,t) & \text{in } D_1, \\ v_t - \big(\mu(x)v_x\big)_x + \alpha(x)v_x + \beta(x)v = g(x,t) & \text{in } D_2, \end{cases} \tag{S_ε}$$

with the initial conditions

$$\begin{cases} u(x,0) = u_0(x), & a \leqslant x \leqslant b, \\ v(x,0) = v_0(x), & b \leqslant x \leqslant c, \end{cases} \tag{IC_ε}$$

boundary conditions

$$\begin{cases} u(a,t) = 0, \\ -v_x(c,t) = \gamma\big(v(c,t)\big), & 0 \leqslant t \leqslant T, \end{cases} \tag{BC_ε}$$

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and transmission conditions at b:

$$\begin{cases} u(b,t) = v(b,t), \\ \varepsilon u_x(b,t) = (\mu v_x)(b,t), \quad 0 \leqslant t \leqslant T. \end{cases} \tag{TC$_\varepsilon$}$$

Here ε is a positive *small* parameter.

The following assumptions will be required in the following:

(I$_1$) $\alpha \in H^1(a,c),\ \beta \in L^\infty(a,c),\ \mu \in H^1(b,c)$;
(I$_2$) $\alpha(x) \geqslant \alpha_0 > 0$ in $[a,c]$, $\mu(x) \geqslant \mu_0 > 0$ in $[b,c]$, $\beta - \alpha'/2 \geqslant 0$ a.e. in (a,c);
(I$_3$) $\gamma : D(\gamma) = \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function.

Obviously, in (I$_2$) we can assume the equivalent conditions $\alpha(x) > 0$ in $[a,c]$ and $\mu(x) > 0$ in $[b,c]$. The constants α_0 and μ_0 are introduced here since we need them later.

Motivation. Problem P$_\varepsilon$ is a model for convection–diffusion processes. Indeed, in some physical problems, the flux of some material may have both a diffusive component and a convective component (due to the flow velocity). If the corresponding domain is one dimensional, say a segment $[a,c]$, then the flux can be expressed as

$$q(x,t) = -\mu w_x + \alpha w,$$

where $w = w(x,t)$ is the density of the material, μ is the diffusion coefficient, and α is the flow velocity. Both these coefficients (μ and α) are assumed to be known and to depend on x only. Moreover, in a subdomain $[a,b]$ the diffusion is considered to be negligible and so in our model we set $\mu(x) = \varepsilon$ for $x \in [a,b]$. Denoting by u and v the restrictions of w to $[a,b]$ and $[b,c]$, respectively, the conservation law leads to a PDE system of the form (S$_\varepsilon$). An additional term depending on $w = (u,v)$ may also occur due to reaction. In our model we assume that the density is constant at $x = a$ and, without any loss of generality, we require that $u(a,t) = 0$. On the other hand, at $x = c$ a nonlinear boundary condition of the form (BC$_\varepsilon$)$_2$ is assumed to be satisfied. This means that the flux depends nonlinearly on the density. The transmission conditions (TC$_\varepsilon$) are quite natural. They express the continuity of w as well as of the flux at the coupling point $x = b$. Certainly, a simpler model corresponding to $\varepsilon = 0$ is preferred. But taking $\varepsilon = 0$ in (S$_\varepsilon$) means a dramatic change of the equation (from parabolic to hyperbolic). It is expected that the solution of the reduced model is discontinuous at $x = b$. On the other hand, it is physically obvious that the flux remains continuous at $x = b$. This fact should be reflected in the corresponding reduced (unperturbed) model. Once we derive such a reduced model (this is done below), we ask ourselves whether this simpler model has a solution which is *close enough* to the solution of the original model P$_\varepsilon$ (that is more realistic since the diffusion in $[a,b]$ is just small, not completely absent). Therefore a mathematical analysis of such problems is extremely important. In particular, this is very useful for the numerical solution. Let us also point out that Gastaldi and Quarteroni [8] discussed the coupling of parabolic and hyperbolic systems as a first step in the numerical treatment of the Navier–Stokes/Euler coupling which is a key issue in Computational Fluid Dynamics.

In [1,2] we studied a similar transmission parabolic–parabolic problem with homogeneous Dirichlet boundary conditions. Here one of the boundary conditions is nonlinear. Such a problem is much more difficult and requires a different specific treatment. We can justify as in [3, Part III, p. 157] (see also [4]) that P$_\varepsilon$ is singularly perturbed, with an internal transition layer located on the left side of the coupling boundary $\Gamma = \{(b,t)\colon\ 0 \leqslant t \leqslant T\}$. In fact, we shall see that the solution of P$_\varepsilon$ has a singular behavior in the vicinity of Γ as ε tends to zero. For details concerning singular perturbation theory we refer the reader, e.g., to the book by Vasilieva, Butuzov and Kalashev [9].

In the present paper we derive an asymptotic expansion of order zero (with respect to ε) of the solution of P_ε. In Section 2 we derive this expansion formally. More precisely, we find out the reduced problem P_0, satisfied by the zero-th order term of the regular series, as well as the problem satisfied by the remainder (of order zero). In addition, we derive an explicit formula for the transition layer correction. In order to validate our asymptotic expansion, we first prove (in Section 3) some results concerning the existence, uniqueness and smoothness of the solutions of P_ε and P_0. Then, in Section 4, we establish some estimates for the remainder components with respect to the uniform convergence norm, which validate completely our expansion. Such an asymptotic analysis is possible under appropriate smoothness and compatibility conditions for the data.

Notice that Section 3 is largely based on the theory of monotone operators and evolution equations in Hilbert spaces. For information in this direction we refer to [5–7].

2. Formal derivation of a zeroth order asymptotic expansion for the solution of P_ε

The classical perturbation theory (see [9] for details) can be adapted to our specific singular perturbation problem, which involves an internal boundary layer. Following this theory, we are going to derive formally an expansion of the solution (u, v) of P_ε of the form:

$$\big(u(x,t,\varepsilon), v(x,t,\varepsilon)\big) = \big(U(x,t), V(x,t)\big) + \big(i(\xi,t), j(\eta,t)\big) + r_\varepsilon(x,t), \tag{2.1}$$

where

$\xi = (b - x)/\varepsilon,\ \eta = (x - b)/\varepsilon$ are the stretched variables corresponding to the two sides of Γ;
(U, V) denotes the zero-th order term of the regular series associated with (u, v);
(i, j) are the boundary (transition) layer corrections of order zero corresponding to the two sides of Γ;
$r_\varepsilon = (r_1, r_2)(x, t, \varepsilon)$ is the remainder of order zero of the expansion (with components corresponding to the two sub-rectangles D_1 and D_2).

Now, we are going to replace (2.1) in the system (S_ε) and then identify the coefficients of $\varepsilon^k,\ k \in \{-2, -1, 0\}$, those depending on x separately from those depending on the fast variables $\xi,\ \eta$. Firstly, we get:

$$\begin{cases} U_t + \alpha U_x + \beta U = f & \text{in } D_1, \\ V_t - (\mu V_x)_x + \alpha V_x + \beta V = g & \text{in } D_2. \end{cases} \tag{S_0}$$

For the correction components we have:

$$\begin{cases} i_{\xi\xi}(\xi,t) + \alpha(b) i_\xi(\xi,t) = 0, \\ \mu(b) j_{\eta\eta}(\eta,t) = 0. \end{cases} \tag{2.2}$$

Since $i,\ j$ are boundary layer functions they should satisfy the following condition at infinity

$$\lim_{\xi\to\infty} i(\xi,t) = \lim_{\eta\to\infty} j(\eta,t) = 0, \quad \forall 0 \leqslant t \leqslant T.$$

So we can deduce from (2.2) that

$$\begin{cases} i(\xi,t) = C(t)\,\mathrm{e}^{-\alpha(b)\xi}, \\ j \equiv 0, \end{cases} \tag{2.3}$$

where the function C will be determined below from (TC_ε). For the components of the remainder we derive the following partial differential system

$$\begin{cases} r_{1t} - \varepsilon r_{1xx} + \alpha r_{1x} + \beta r_1 = \varepsilon U_{xx} - h_\varepsilon \quad \text{in } D_1, \\ r_{2t} - (\mu r_{2x})_x + \alpha r_{2x} + \beta r_2 = 0 \quad \text{in } D_2, \end{cases} \tag{SR}$$

where

$$h_\varepsilon(x,t) := i_t + \beta i + \big(\alpha(x) - \alpha(b)\big) i_x \quad \text{in } D_1.$$

From (IC_ε) and (BC_ε) it follows

$$\begin{cases} U(x,0) = u_0(x), \quad x \in [a,b], \\ V(x,0) = v_0(x), \quad x \in [b,c], \end{cases} \tag{IC_0}$$

$$i(\xi,0) = 0 \Leftrightarrow C(0) = 0, \tag{2.4}$$

$$\begin{cases} r_1(x,0,\varepsilon) = 0, \quad a \leqslant x \leqslant b, \\ r_2(x,0,\varepsilon) = 0, \quad b \leqslant x \leqslant c, \end{cases} \tag{ICR}$$

as well as

$$\begin{cases} U(a,t) = 0, \\ -V_x(c,t) = \gamma\big(V(c,t)\big), \quad 0 \leqslant t \leqslant T, \end{cases} \tag{BC_0}$$

$$\begin{cases} r_1(a,t,\varepsilon) = -i\big(\xi(a),t\big), \\ -r_{2x}(c,t,\varepsilon) = \gamma\big(V(c,t) + r_2(c,t,\varepsilon)\big) - \gamma\big(V(c,t)\big), \quad 0 \leqslant t \leqslant T, \end{cases} \tag{BCR}$$

with $\xi(a) = (b-a)/\varepsilon$.

Finally, by using (TC_ε) we get

$$C(t) = V(b,t) - U(b,t), \tag{2.5}$$

$$(\mu V_x)(b,t) = \alpha(b)\big(V(b,t) - U(b,t)\big), \quad 0 \leqslant t \leqslant T, \tag{TC_0}$$

$$\begin{cases} r_1(b,t,\varepsilon) = r_2(b,t,\varepsilon), \\ \varepsilon r_{1x}(b,t,\varepsilon) - (\mu r_{2x})(b,t,\varepsilon) = -\varepsilon U_x(b,t), \quad 0 \leqslant t \leqslant T. \end{cases} \tag{TCR}$$

Note that the condition $C(0) = 0$ (see (2.4) above) will also occur later as a compatibility condition satisfied by the data in order to achieve enough smoothness.

Summarizing, we can see that (U, V) satisfies the reduced problem P_0 which is made up by (S_0), (IC_0), (BC_0), (TC_0), while the pair (r_1, r_2) satisfies the problem (SR), (ICR), (BCR), (TCR), which will be called R. This is of the same type as the original problem P_ε. On the other hand, P_0 is a coupled hyperbolic–parabolic problem. The transmission condition (TC_0) was discussed in [3,4]. It can also be obtained by a variational approach (see [8]), so it is quite natural.

3. Existence, uniqueness and smoothness of the solutions of the problems $\mathbf{P_0}$ and $\mathbf{P_\varepsilon}$

In order to investigate the problem P_ε, we consider the real product space $H := L^2(a,b) \times L^2(b,c)$. This is a Hilbert space with respect to the usual scalar product, denoted $\langle\cdot,\cdot\rangle$ and the corresponding induced norm, denoted $\|\cdot\|$. Now, we define the operator $J_\varepsilon : D(L_\varepsilon) \subset H \to H$ as follows

$$\begin{aligned} D(J_\varepsilon) &:= \bigl\{(p,q) \in H^2(a,b) \times H^2(b,c),\ p(a) = 0,\ p(b) = q(b), \\ &\qquad \varepsilon p'(b) = (\mu q')(b),\ -q'(c) = \gamma\bigl(q(c)\bigr)\bigr\}, \\ J_\varepsilon(p,q) &:= \bigl(-\varepsilon p'' + \alpha p' + \beta p,\ -(\mu q')' + \alpha q' + \beta q\bigr). \end{aligned}$$

Concerning this operator we have the following result:

Lemma 3.1. *If* (I_1)–(I_3) *hold then the operator* J_ε *defined above is maximal monotone for every* $\varepsilon > 0$.

Proof. Taking into account (I_1)–(I_3) it is easily seen that J_ε is monotone. It is also easy to show that $D(J_\varepsilon)$ is a nonempty set. In fact, this also follows from the next step of our proof. So now let us prove that J_ε is maximal monotone, i.e., for every $(f_1, f_2) \in H$ there exists $(p,q) \in D(J_\varepsilon)$ such that

$$(p,q) + J_\varepsilon(p,q) = (f_1, f_2). \tag{3.1}$$

Let $(\tilde{p}, \tilde{q}) \in H^2(a,b) \times H^2(b,c)$ be the unique solution of Eq. (3.1) satisfying the following conditions

$$\begin{cases} \tilde{p}(a) = 0, \quad \tilde{p}(b) = \tilde{q}(b), \\ \varepsilon\tilde{p}'(b) = \mu(b)\tilde{q}'(b), \quad \tilde{q}(c) = 0. \end{cases}$$

The existence and uniqueness of $(\tilde{p}, \tilde{q})$ is known (see [3, p. 160]). In fact, the key argument is based on the Lax–Milgram theorem. Now, $(p,q) + (\tilde{p}, \tilde{q})$ is a solution of Eq. (3.1) if and only if (p,q) satisfies the homogeneous equation

$$(p,q) + J_\varepsilon(p,q) = (0,0) \tag{3.2}$$

as well as the following boundary and transmission conditions

$$\begin{cases} p(a) = 0, \quad p(b) = q(b), \quad \varepsilon p'(b) = (\mu q')(b), \\ q'(c) + \gamma\bigl(q(c)\bigr) = -\tilde{q}'(c). \end{cases} \tag{3.3}$$

Let $p_1, p_2 \in H^2(a,b)$, $q_1, q_2 \in H^2(b,c)$ be some fundamental systems of real solutions for the two ordinary differential equations of (3.2). So the general solution of (3.2) is given by

$$p = c_1 p_1 + c_2 p_2, \qquad q = d_1 q_1 + d_2 q_2, \tag{3.4}$$

where $c_1,\ c_2,\ d_1,\ d_2$ are real constants. If we replace (3.4) in (3.3) we get an algebraic system in $c_1,\ c_2,\ c_3,\ c_4$. So, in fact it suffices to show that this system has a solution. We can simplify this algebraic system. Indeed, we can assume that $p_1(a) = 0,\ p_2(a) = 1,\ q_1(c) = 0,\ q_1'(c) = 1,\ q_2(c) = 1,\ q_2'(c) = 0$; then $q'(c) = d_1$ and the condition $p(a) = 0$ implies $c_2 = 0$, so we arrive at the following system in $c_1,\ d_1,\ d_2$:

$$\begin{cases} p_1(b)c_1 - q_1(b)d_1 = q_2(b)d_2, \\ \varepsilon p_1'(b)c_1 - \mu(b)q_1'(b)d_1 = \mu(b)q_2'(b)d_2, \\ d_1 + \gamma(d_2) = -\tilde{q}'(c). \end{cases} \tag{3.5}$$

Note that the first two equations of (3.5) form a linear system, say (LS), which is uniquely solvable with respect to $c_1,\ d_1$. This follows from the fact that the homogeneous system associated with (LS) (which is obtained by taking $d_2 = 0$) has the null solution only. Indeed, $(p = c_1p_1,\ q = d_1q_1)$ (with $c_1,\ d_1$ satisfying this homogeneous system) is a solution of Eq. (3.2) with the homogeneous Dirichlet boundary conditions and transmission conditions, which implies (by uniqueness) that $p = 0,\ q = 0$, i.e., $c_1 = d_1 = 0$, as asserted. Now, as we can solve (LS) with respect to $c_1,\ d_1$, taking into account $(3.5)_3$, we can reduce our problem to a nonlinear algebraic equation in d_2:

$$\lambda d_2 + \gamma(d_2) = -\tilde{q}'(c). \tag{3.6}$$

We will not use the explicit expression of the constant λ. In order to show that Eq. (3.6) has a solution, it suffices to show that $\lambda > 0$ or, equivalently,

$$q'(c) \cdot d_2 > 0, \quad \forall d_2 \neq 0.$$

Indeed, for $p = c_1p_1,\ q = d_1q_1 + d_2q_2$ (with $c_1,\ d_1$ satisfying $(3.5)_1$, $(3.5)_2$), we have

$$0 = \langle (p,q) + J_\varepsilon(p,q), (p,q)\rangle \geqslant \|(p,q)\|^2 - \mu(c)q'(c)q(c) + \frac{1}{2}\alpha(c)q(c)^2.$$

Therefore,

$$\mu(c)q'(c)d_2 \geqslant \|(p,q)\|^2 > 0 \quad \forall d_2 \neq 0. \quad \square$$

Notice that the problem P_ε can be expressed as the following Cauchy problem in H:

$$\begin{cases} W_\varepsilon'(t) + J_\varepsilon W_\varepsilon(t) = F(t), \quad 0 < t < T, \\ W_\varepsilon(0) = W_0, \end{cases} \tag{3.7}$$

where $W_\varepsilon(t) := (u(\cdot,t,\varepsilon), v(\cdot,t,\varepsilon)),\ W_0 := (u_0, v_0),\ F(t) := (f(\cdot,t), g(\cdot,t))$.

We have the following result, whose proof is essentially known (see [1], [3, p. 177]):

Proposition 3.1. *If our assumptions* (I_1)–(I_3) *hold and, in addition,*

$$F \in W^{1,1}(0,T;H), \tag{3.8}$$

$$W_0 \in D(J_\varepsilon), \tag{3.9}$$

then, for every $\varepsilon > 0$ *the problem* (3.7) *has a unique strong solution*

$$W_\varepsilon \in W^{1,\infty}(0,T;H) \cap W^{1,2}\big(0,T;H^1(a,b) \times H^1(b,c)\big) \cap L^\infty\big(0,T;H^2(a,b) \times H^2(b,c)\big).$$

Notice that the problem (3.7) is nonlinear. However, no essential difference appears as compared to the linear case in [1,3]. So we leave to the reader the proof of this theorem.

Remark 3.1. The conditions (3.8), (3.9) hold if and only if

$$\begin{cases} f \in W^{1,1}\big(0,T;L^2(a,b)\big), \quad g \in W^{1,1}\big(0,T;L^2(b,c)\big), \\ u_0 \in H^2(a,b), \quad v_0 \in H^2(b,c), \\ u_0(a) = 0, \quad u_0(b) = v_0(b), \\ \varepsilon u_0'(b) = \mu(b) v_0'(b), \quad -v_0'(c) = \gamma(v_0(c)). \end{cases} \tag{3.10}$$

By (3.10) we can see that $C(0) = 0$ (see (2.4)), i.e., this is a compatibility condition.

It is important to note that we want Proposition 3.1 to be valid for all $\varepsilon > 0$ and so we have to require in addition (see (3.10)):

$$u_0'(b) = 0, \qquad v_0'(b) = 0. \tag{3.11}$$

In the following we are going to investigate the reduced problem P_0. In fact, we can split it into two separate problems, with the unknowns U and V, respectively. In order to study the problem satisfied by U, we need the real Hilbert space $H_1 := L^2(a,b)$, which is equipped with the usual scalar product, denoted $\langle\cdot,\cdot\rangle_1$, and the induced norm, denoted $\|\cdot\|_1$. Define the operator $A\colon D(A) \subset H_1 \to H_1$,

$$D(A) := \{p \in H^1(a,b)\colon\ p(a) = 0\}, \qquad A(p) := \alpha p' + \beta p.$$

Under (I_1)–(I_3) this operator is linear maximal monotone. It is just an easy matter to show this.

Clearly, the problem satisfied by U can be written as the following initial value problem in H_1:

$$\begin{cases} U'(t) + AU(t) = f(t), \\ U(0) = u_0, \end{cases} \tag{3.12}$$

where $U(t) := U(\cdot,t)$, $f(t) := f(\cdot,t)$.

We have the following result:

Proposition 3.2. *If* (I_1)–(I_3) *holds and, in addition,*

$$\alpha, \beta \in W^{1,\infty}(a,b), \tag{3.13}$$

$$f \in W^{2,1}(0,T;H_1) \cap L^\infty\big(0,T;H^1(a,b)\big), \tag{3.14}$$

$$u_0, f(\cdot,0) - A(u_0) \in D(A), \tag{3.15}$$

then the problem (3.12) *admits a unique strong solution*

$$U \in W^{1,\infty}\big(0,T;H^1(a,b)\big) \cap W^{2,\infty}(0,T;H_1) \cap L^\infty\big(0,T;H^2(a,b)\big).$$

The proof is essentially contained in [2].

Remark 3.2. The conditions (3.14), (3.15) are satisfied if, in addition to the assumptions of Proposition 3.2, we have

$$\begin{cases} f \in W^{2,\infty}(0,T;L^2(a,b)), \quad f(\cdot,0) \in H^1(a,b), \quad u_0 \in H^2(a,b), \\ u_0(a) = 0, \quad \alpha(a)u_0'(a) + \beta(a)u_0(a) - f(a,0) = 0. \end{cases} \tag{3.16}$$

Finally, we are going to investigate the problem satisfied by V. To this purpose, we consider the Hilbert space $H_2 := L^2(b,c)$, endowed with the usual scalar product and norm, denoted by $\langle \cdot,\cdot \rangle_2$ and $\|\cdot\|_2$. Define the operator $B : D(B) \subset H_2 \to H_2$ as follows:

$$D(B) := \{p \in H^2(b,c)\colon \mu(b)p'(b) = \alpha(b)p(b),\ -p'(c) = \gamma(p(c))\},$$
$$B(p) := -(\mu p')' + \alpha p' + \beta p.$$

Lemma 3.2. *If the assumptions* (I_1)–(I_3) *hold, then the operator* $B : D(B) \to H_2$ *defined above is maximal monotone.*

Proof. An easy computation shows that B is monotone. It remains to show its maximality. Equivalently, this means that for every $f \in H_2$ there exists a $p \in D(B)$ such that

$$p + B(p) = f. \tag{3.17}$$

In particular this will also imply that $D(B)$ is a nonempty set. Let $\tilde{p} \in H^2(b,c)$ be a solution of Eq. (3.17) satisfying the boundary conditions

$$\alpha(b)\tilde{p}(b) = \mu(b)\tilde{p}'(b), \qquad \tilde{p}(c) = 0.$$

We know that such a solution exists and is unique [3, p. 161]. Obviously, $p + \tilde{p}$ is a solution of (3.17) if and only if

$$\begin{cases} p + Bp = 0, \\ (\mu p')(b) = \alpha(b)p(b), \quad -\tilde{p}'(c) - p'(c) = \gamma(p(c)). \end{cases} \tag{3.18}$$

Let (p_1,p_2) be the fundamental system of solutions for the homogeneous equation $(3.18)_1$ satisfying $p_1(c) = 1,\ p_1'(c) = 0,\ p_2(c) = 0,\ p_2'(c) = 1$. So the general solution of $(3.18)_1$ is given by

$$p = c_1p_1 + c_2p_2, \tag{3.19}$$

where $c_1,\ c_2$ are real constants. If we replace (3.19) in $(3.18)_2$, we arrive at the algebraic system

$$\begin{cases} [\alpha(b)p_2(b) - \mu(b)p_2'(b)]c_2 = [\mu(b)p_1'(b) - \alpha(b)p_1(b)]c_1, \\ c_2 + \gamma(c_1) = -\tilde{p}'(c). \end{cases} \tag{3.20}$$

It suffices to show that this system has a solution. It is easy to see that $\alpha(b)p_2(b) - \mu(b)p_2'(b) \neq 0$. Indeed, if we assume by contradiction that $\alpha(b)p_2(b) - \mu(b)p_2'(b) = 0$, then $p := p_2$ satisfies the homogeneous equation $(3.18)_1$ as well as the boundary conditions $\alpha(b)p(b) = \mu(b)p'(b)$, $p(c) = 0$ and so (by uniqueness) p_2 is the null function, which is impossible. So, we can solve Eq. $(3.20)_1$ with respect to c_2 thus arriving at the equation:

$$\lambda c_1 + \gamma(c_1) = -\tilde{p}'(c). \tag{3.21}$$

One can show that $\lambda > 0$, which implies that (3.21) has a (unique) solution. To this purpose, we can use the same argument as in Lemma 3.1. So, for a function p given by (3.19), with c_1, c_2 satisfying $(3.20)_1$, we have

$$\begin{aligned} 0 = \langle p + Bp, p\rangle_2 &= \|p\|_2^2 - \mu p'p|_b^c + \frac{1}{2}\alpha p^2|_b^c + \int_b^c \left\{\mu(p')^2 + \left(\beta - \frac{\alpha'}{2}\right)p^2\right\} \mathrm{d}x \\ &\geqslant \|p\|_2^2 - \mu(c)p'(c)p(c) + \frac{1}{2}\alpha(c)p(c)^2 + \frac{1}{2}\alpha(b)p(b)^2, \end{aligned}$$

and hence

$$\mu(c)\lambda c_1^2 = \mu(c)p'(c)p(c) \geqslant \|p\|_2^2 > 0 \quad \forall c_1 \neq 0,$$

which shows that $\lambda > 0$, as asserted. □

In the following we consider the problem

$$z_t - (\mu z_x)_x + \alpha z_x + \beta z = \zeta(x,t) \quad \text{in } D_2, \tag{3.22}$$

$$z(x,0) = z_0(x), \quad x \in [b,c], \tag{3.23}$$

$$\alpha(b)z(b,t) + \theta(t) = \mu(b)z_x(b,t), \quad 0 \leqslant t \leqslant T, \tag{3.24}$$

$$-z_x(c,t) = \gamma\big(z(c,t)\big), \quad 0 \leqslant t \leqslant T, \tag{3.25}$$

for which we are able to prove the following result:

Proposition 3.3. *If* (I_1)–(I_3) *are fulfilled and, in addition,*

$$\zeta \in W^{1,1}\big(0,T;L^2(b,c)\big), \qquad z_0 \in H^2(b,c), \tag{3.26}$$

$$\begin{cases} \alpha(b)z_0(b) + \theta(0) = (\mu z_0')(b), \\ -z_0'(c) = \gamma\big(z_0(c)\big), \end{cases} \tag{3.27}$$

$$\theta \in W^{1,2}(0,T), \tag{3.28}$$

then the problem (3.22)–(3.25) *has a unique strong solution*

$$z \in W^{1,\infty}\big(0,T;L^2(b,c)\big) \cap W^{1,2}\big(0,T;H^1(b,c)\big) \cap L^\infty\big(0,T;H^2(b,c)\big).$$

Proof. In the first step of the proof we assume $\theta \in W^{2,1}(0,T)$. In order to homogenize (3.24), we consider the substitution

$$\bar{z}(x,t) = z(x,t) + M(t)x^2 + N(t)x + P(t),$$

where

$$M(t) = \frac{\theta(t)}{\alpha(b)(c-b)^2 + 2\mu(b)(c-b)}, \qquad N(t) = -2cM(t), \qquad P(t) = c^2 M(t).$$

It is easily seen that $Z(t) := \bar{z}(\cdot,t)$ satisfies the following Cauchy problem in H_2:

$$\begin{cases} Z'(t) + BZ(t) = S(t), \quad 0 < t < T, \\ Z(0) = Z_0, \end{cases} \tag{3.29}$$

where $Z_0 = \bar{z}(\cdot,0)$, $S(t) = \bar{\zeta}(\cdot,t)$,

$$\begin{aligned} \bar{\zeta}(x,t) = {} & \zeta(x,t) + M'(t)x^2 + N'(t)x + P'(t) - 2\mu'(x)xM(t) - \mu'(x)N(t) \\ & - 2\mu(x)M(t) + \alpha(x)\big[2M(t)x + N(t)\big] + \beta(x)\big[M(t)x^2 + N(t)x + P(t)\big] \quad \text{in } D_2, \\ Z_0(x) = {} & z_0(x) + M(0)x^2 + N(0)x + P(0), \quad x \in [b,c]. \end{aligned}$$

Taking into account (3.26), (3.27) as well as our temporary assumption $\theta \in W^{2,1}(0,T)$, we get

$$S \in W^{1,1}(0,T;H_2), \qquad Z_0 \in D(B),$$

which in virtue of the standard existence theory (see, e.g., [7, p. 48]) implies that the problem (3.29) admits a unique strong solution $Z \in W^{1,\infty}(0,T;H_2)$. Now, by the obvious estimate

$$\begin{aligned} & \mu_0 \int_b^c \big(\bar{z}_x(x,t) - \bar{z}_0'(x)\big)^2 \,\mathrm{d}x \\ & \quad \leqslant \int_b^c \big(\bar{\zeta}(x,t) - \bar{z}_t(x,t) - BZ_0(x)\big)\big(\bar{z}(x,t) - \bar{z}_0(x)\big) \leqslant \mathit{Const.}, \quad \text{a.a. } t \in (0,T), \end{aligned}$$

it follows that $\bar{z}_x \in L^\infty(0,T;H^1(b,c))$. So, $(3.29)_1$ yields $\bar{z} \in L^\infty(0,T;H^2(b,c))$. In order to show that $Z \in W^{1,2}(0,T;H^1(b,c))$, we write $(3.29)_1$ in $t, t+h \in [0,T]$, where $h > 0$ is assumed to be sufficiently small, and then we multiply in H_2 the difference of the two equations by $Z(t+h) - Z(t)$:

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|Z(t+h) - Z(t)\big\|_2^2 + \mu_0 \int_b^c \big(\bar{z}_x(x,t+h) - \bar{z}_x(x,t)\big)^2 \,\mathrm{d}x \\ & \quad \leqslant \big\|Z(t+h) - Z(t)\big\|_2 \big\|S(t+h) - S(t)\big\|_2. \end{aligned}$$

This implies

$$\int_0^{T-h} \big\|\bar{z}_x(\cdot,t+h) - \bar{z}_x(\cdot,t)\big\|_2^2 \,\mathrm{d}t \leqslant Ch^2,$$

and hence $\bar{z}_x \in W^{1,2}(0,T;H_2)$. We have used the condition $S \in W^{1,1}(0,T;H_2)$ as well as the fact that Z is Lipschitzian from $[0,T]$ to H_2.

Finally, the case $\theta \in W^{1,2}(0,T)$ can be solved by a density argument. Consider a sequence $\{\theta_n\}_n \subseteq W^{2,1}(0,T)$ such that $\theta_n \to \theta$ in $H^1(0,T)$. We denote by $\{Z_n\}_n$ the sequence of solutions of (3.29) corresponding to $\theta := \theta_n$. According to the first step, Z_n exists for every n and has the regularity indicated in the statement of the theorem. By a standard argument, we can show that $Z_n \to Z$ in $W^{1,\infty}(0,T;L^2(b,c)) \cap W^{1,2}(0,T;H^1(b,c))$, from which our conclusion follows. □

Remark 3.3. Since V satisfies a problem of the type (3.22)–(3.25) with $\theta(t) = -\alpha(b)U(b,t),\ 0 \leqslant t \leqslant T,\ z_0 = v_0,\ \zeta = g$, we deduce from Proposition 3.3 that

$$V \in W^{1,\infty}\big(0,T;L^2(b,c)\big) \cap W^{1,2}\big(0,T;H^1(b,c)\big) \cap L^\infty\big(0,T;H^2(b,c)\big),$$

whenever the following assumptions are fulfilled:

$$\begin{cases} g \in W^{1,1}\big(0,T;L^2(b,c)\big), \quad U(b,\cdot) \in W^{1,2}(0,T), \quad v_0 \in H^2(b,c), \\ \alpha(b)\big(v_0(b) - u_0(b)\big) = (\mu v_0')(b), \\ -v_0'(c) = \gamma\big(v_0(c)\big). \end{cases} \tag{3.30}$$

By virtue of what we have proved so far (see especially Remarks 3.1–3.3), we can formulate the following concluding result:

Corollary 3.1. *Assume that* (I_1)–(I_3) *hold,*

$$f \in W^{2,1}\big(0,T;L^2(a,b)\big) \cap L^\infty\big(0,T;H^1(a,b)\big), \quad f(\cdot,0) \in H^1(a,b), \quad g \in W^{1,1}\big(0,T;L^2(b,c)\big),$$
$$\alpha|_{[a,b]}, \beta|_{[a,b]} \in W^{1,\infty}(a,b), \quad u_0 \in H^2(a,b), \quad v_0 \in H^2(b,c),$$

and, in addition, the following compatibility conditions are fulfilled

$$\begin{cases} u_0(a) = 0, \quad u_0(b) = v_0(b), \quad u_0'(b) = v_0'(b) = 0, \\ -v_0'(c) = \gamma\big(v_0(c)\big), \quad \alpha(b)\big(v_0(b) - u_0(b)\big) = \mu(b)v_0'(b), \\ \alpha(a)u_0'(a) - f(a,0) = 0. \end{cases} \tag{3.31}$$

Then, for each $\varepsilon > 0$, P_ε *has a unique strong solution*

$$(u,v) \in W^{1,\infty}\big(0,T;L^2(a,b) \times L^2(b,c)\big) \cap W^{1,2}\big(0,T;H^1(a,b) \times H^1(b,c)\big)$$
$$\cap L^\infty\big(0,T;H^2(a,b) \times H^2(b,c)\big),$$

while the problem P_0 *admits a unique strong solution,*

$$(U,V) \in \big(W^{1,\infty}\big(0,T;H^1(a,b)\big) \cap W^{2,\infty}\big(0,T;L^2(a,b)\big) \cap L^\infty\big(0,T;H^2(a,b)\big)\big)$$
$$\times \big(W^{1,\infty}\big(0,T;L^2(b,c)\big) \cap W^{1,2}\big(0,T;H^1(b,c)\big) \cap L^\infty\big(0,T;H^2(b,c)\big)\big).$$

4. Estimates for the remainder components

Under the assumptions of Corollary 3.1, the expansion (2.1) is well defined, in the sense that all its terms exist. We are now going to show that this is a real asymptotic expansion, that is the remainder "approaches" zero. More precisely, we have:

Theorem 4.1. *If all the assumptions of Corollary* 3.1 *are fulfilled, then the solution of* P_ε *has an expansion of the form* (2.1)*, where the remainder components satisfy the estimates*

$$\|r_1(\cdot,\cdot,\varepsilon)\|_{C(\overline{D}_1)} = \mathrm{O}(\varepsilon^{1/8}), \tag{4.1}$$

$$\|r_2(\cdot,\cdot,\varepsilon)\|_{C(\overline{D}_2)} = \mathrm{O}(\varepsilon^{3/8}). \tag{4.2}$$

Proof. From Corollary 3.1 we deduce that $r_\varepsilon(t) := (r_1(\cdot,t,\varepsilon), r_2(\cdot,t,\varepsilon))$ satisfies

$$r_\varepsilon \in W^{1,2}\big(0,T;H^1(a,b)\times H^1(b,c)\big)\cap L^\infty\big(0,T;H^2(a,b)\times H^2(b,c)\big),$$

that is r_ε is a strong (differentiable a.e.) solution of the problem R. Now, we multiply (SR) in H by r_ε and integrate over $[0,t]$ thus obtaining:

$$\begin{aligned}
&\frac{1}{2}\big\|r_\varepsilon(t)\big\|^2 + \varepsilon\|r_{1x}\|^2_{L^2((a,b)\times(0,t))} + \mu_0\|r_{2x}\|^2_{L^2((b,c)\times(0,t))}\\
&\quad\leqslant \varepsilon\int_0^t \big|U_x(b,s)r_1(b,s,\varepsilon)\big|\,\mathrm{d}s + \varepsilon\int_0^t \big|r_{1x}(a,s,\varepsilon)\big|\cdot\big|i\big(\xi(a),s\big)\big|\,\mathrm{d}s\\
&\qquad + \frac{\alpha(a)}{2}\int_0^t i\big(\xi(a),s\big)^2\,\mathrm{d}s + \int_0^t \big\|\omega_\varepsilon(\cdot,s)\big\|_1\big\|r_1(\cdot,s,\varepsilon)\big\|_1\,\mathrm{d}s,
\end{aligned} \tag{4.3}$$

where

$$\omega_\varepsilon = \varepsilon U_{xx}(x,t) - i_t(\xi,t) - \beta(x)i(\xi,t) - \big(\alpha(x)-\alpha(b)\big)i_x(\xi,t), (x,t)\in(a,b)\times(0,T).$$

Now, as W_ε is a strong solution of the problem (3.8), (3.9), we have the classical estimate (see, e.g., [7, p. 48])

$$\big\|W_\varepsilon'(t)\big\| \leqslant \big\|F(0) - J_\varepsilon W_0\big\| + \int_0^t \big\|F'(s)\big\|\,\mathrm{d}s = \mathrm{O}(1), \tag{4.4}$$

and hence

$$\big\|W_\varepsilon(t)\big\| = \left\|W_0 + \int_0^t W_\varepsilon'(s)\,\mathrm{d}s\right\| \leqslant \|W_0\| + \int_0^t \big\|W_\varepsilon'(s)\big\|\,\mathrm{d}s = \mathrm{O}(1). \tag{4.5}$$

From the obvious equation

$$\big\langle W_\varepsilon'(t), W_\varepsilon(t)-W_0\big\rangle + \big\langle J_\varepsilon(W_\varepsilon(t)-W_0), W_\varepsilon(t)-W_0\big\rangle = \big\langle F(t)-J_\varepsilon W_0, W_\varepsilon(t)-W_0\big\rangle$$

we get

$$\varepsilon\|u_x(\cdot,t,\varepsilon)-u_0'\|_1^2+\mu_0\|v_x(\cdot,t,\varepsilon)-v_0'\|_2^2\leqslant\|W_\varepsilon(t)-W_0\|\big(\|F(t)\|+\|J_\varepsilon W_0\|+\|W_\varepsilon'(t)\|\big)$$
$$=\mathrm{O}(1). \tag{4.6}$$

In particular,

$$\|v_x(\cdot,t,\varepsilon)\|_2=\mathrm{O}(1), \tag{4.7}$$

and this together with (4.5) implies that

$$\|v(\cdot,\cdot,\varepsilon)\|_{C([b,c]\times[0,T])}=\mathrm{O}(1). \tag{4.8}$$

We have used the fact that $H^1(b,c)$ is continuously embedded into $C[b,c]$. By (4.8) and $(\mathrm{TC}_\varepsilon)_1$ it follows that

$$\|u(b,\cdot,\varepsilon)\|_{C[0,T]}=\mathrm{O}(1). \tag{4.9}$$

Using (2.1) and (4.9) we can see that

$$\|r_1(b,\cdot,\varepsilon)\|_{C[0,T]}=\mathrm{O}(1). \tag{4.10}$$

This boundedness will be used to estimate the first term of the right-hand side of (4.3) and we are going to investigate the other terms there. To this purpose, we note that (see (4.8) and $(\mathrm{BC}_\varepsilon)_2$)

$$\|v_x(c,\cdot,\varepsilon)\|_{L^\infty(0,T)}=\mathrm{O}(1). \tag{4.11}$$

Now, we integrate with respect to x the two equations of (S_ε) over $[a,b]$ and $[b,c]$, respectively. Then, we add the resulting equations and, by taking account of $(\mathrm{TC}_\varepsilon)_2$, (4.4)–(4.6), and (4.11), we get

$$\|u_x(a,\cdot,\varepsilon)\|_{L^\infty(0,T)}=\mathrm{O}\big(\varepsilon^{-3/2}\big). \tag{4.12}$$

Clearly, by (2.1) and (4.12), we have

$$\|r_{1x}(a,\cdot,\varepsilon)\|_{L^\infty(0,T)}=\mathrm{O}\big(\varepsilon^{-3/2}\big). \tag{4.13}$$

If we take a look at the structure of ω_ε, we can easily see that

$$\|\omega_\varepsilon\|_{L^2((a,b)\times(0,T))}=\mathrm{O}\big(\varepsilon^{1/2}\big). \tag{4.14}$$

By (4.3), (4.10) and (4.13) there exists a constant $M>0$ such that

$$\frac{1}{2}\|r_\varepsilon(t)\|^2\leqslant M\varepsilon+\int_0^t\|\omega_\varepsilon(\cdot,s)\|_1\|r_\varepsilon(s)\|\,\mathrm{d}s,$$

which by (4.14) and by a Gronwall type lemma (see [5, p. 156]) leads to

$$\|r_\varepsilon\|_{C([0,T];L^2(a,b)\times L^2(b,c))} = \mathrm{O}(\varepsilon^{1/2}). \tag{4.15}$$

Now, if we take the scalar product in H of (r_1, r_2) with the system

$$\begin{cases} (u-U)_t - \varepsilon r_{1xx} + \alpha r_{1x} + \beta r_1 = \rho_\varepsilon & \text{in } D_1, \\ r_{2t} - (\mu r_{2x})_x + \alpha r_{2x} + \beta r_2 = 0 & \text{in } D_2, \end{cases}$$

where $\rho_\varepsilon(x,t) = \varepsilon U_{xx} - \beta i + (\alpha(b) - \alpha(x)) i_x$, we get after an easy computation

$$\begin{aligned} \varepsilon\|r_{1x}(\cdot,t,\varepsilon)\|_1^2 + \mu_0\|r_{2x}(\cdot,t,\varepsilon)\|_2^2 \leqslant M_1\|r_\varepsilon(\cdot,t)\| + \|\rho_\varepsilon(\cdot,t)\|_1\|r_1(\cdot,t,\varepsilon)\|_1 + \varepsilon|r_1(b,t,\varepsilon)U_x(b,t)| \\ + \varepsilon|r_{1x}(a,t,\varepsilon)i(\xi(a),t)| + \frac{1}{2}\alpha(a)i^2(\xi(a),t), \end{aligned} \tag{4.16}$$

where M_1 is a positive constant. Taking into account (4.10), (4.13), (4.15) and the obvious estimate $\|\rho_\varepsilon(\cdot,\cdot,\varepsilon)\|_{L^\infty(0,T;L^2(a,b))} = \mathrm{O}(\varepsilon^{1/2})$, we can deduce from (4.16) that

$$\|r_{1x}(\cdot,\cdot,\varepsilon)\|_{L^\infty(0,T;L^2(a,b))} = \mathrm{O}(\varepsilon^{-1/4}), \tag{4.17}$$

$$\|r_{2x}(\cdot,\cdot,\varepsilon)\|_{L^\infty(0,T;L^2(b,c))} = \mathrm{O}(\varepsilon^{1/4}). \tag{4.18}$$

In order to prove (4.1), we can use the formula

$$r_1^2(x,t,\varepsilon) = 2\int_a^x r_1(y,t,\varepsilon) r_{1y}(y,t,\varepsilon)\,\mathrm{d}y + r_1^2(a,t,\varepsilon)$$

together with (4.15) and (4.17). This formula is convenient since $r_1(a,t,\varepsilon) = -i(\xi(a),t)$ converges exponentially to zero as ε tends to zero, uniformly with respect to $t \in [0,T]$. Finally, to prove the estimate (4.2) we can use a similar argument. By (4.15) and the mean value theorem we can associate with every (t,ε) some $y_{t\varepsilon} \in [b,c]$ such that $|r_2(y_{t\varepsilon},t,\varepsilon)| = \mathrm{O}(\varepsilon^{1/2})$. This together with the formula

$$r_2^2(x,t,\varepsilon) = 2\int_{y_{t\varepsilon}}^x r_2(y,t,\varepsilon) r_{2y}(y,t,\varepsilon)\,\mathrm{d}y + r_2^2(y_{t\varepsilon},t,\varepsilon)$$

implies (4.1) (cf. (4.15) and (4.18)). The proof is now complete.

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