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Existence and high regularity of the solution of a nonlinear parabolic problem with algebraic-differential boundary conditions

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Dedicated to Professor Viorel Barbu on the ocassion of his 60^{th} *birthday*

In this paper we investigate a nonlinear 1D parabolic problem with algebraic-differential boundary conditions. Existence, uniqueness and higher regularity of the solution is proved. It is shown that actually any regularity can be obtained provided that appropriate smoothness of the data and compatibility assumptions are required.

1 Introduction

The aim of this paper is to study nonlinear parabolic boundary value problems (BVP) of the form

$$y_t - y_{xx} + g(y) \ni f(x,t) \quad \text{in} \quad D_T\,, \tag{E}$$

$$\begin{cases} -y_x(0,t) + \beta_1(y(0,t)) \ni s_1(t)\,, \\ y_t(1,t) + y_x(1,t) + \beta_2(y(1,t)) \ni s_2(t) \quad \text{for} \quad 0 < t < T\,, \end{cases} \tag{BC}$$

$$y(x,0) \;=\; y_0(x) \quad \text{for} \quad 0 \;<\; x \;<\; 1\,. \tag{IC}$$

Here D_T denotes the rectangle $(0,1)\times(0,T)$ for fixed $T \in (0,\infty)$. The nonlinear mappings $g : D(g) \subset \mathbb{R} \to \mathbb{R}$ and $\beta_i : D(\beta_i) \subset \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are possibly multivalued, while $f,\ s_1,\ s_2,\ y_0$ are given data.

Note that the first boundary condition, $(\mathrm{BC})_1$, is an algebraic relation between $y(0,t)$, $y_x(0,t)$ and $s_1(t)$ while $(\mathrm{BC})_2$ is a differential equation with respect to t since the time derivative $y_t(1,t)$ is here involved. This is why we call (BC) *algebraic-differential boundary conditions*.

A similar parabolic equation, but with both boundary conditions of algebraic type, was recently investigated in [7]. However, the present (BVP) is essentially different and is requiring a separate analysis.

Our (BVP) serves as a model for diffusion in chemical substances where the boundary condition $(\mathrm{BC})_2$ describes a reaction at the boundary. The term $y_x(1,t)$ is responsible for the diffusive transport of materials to the boundary (see [10] for the linear case of the (BVP)). Also such a problem may apear as a reduced (unperturbed) model for the telegraph boundary value problem with algebraic-differential boundary conditions, where the distributed specific inductance is negligible and, therefore, is modeled as a small parameter (see [2], [4] [5], [17]). Among other topics, we extend here some previous results concerning particular cases of the (BVP), included in the papers just quoted before. There, $(\mathrm{BC})_1$ was linear which allowed us to use the variation of constants formula associated with the corresponding evolution equation. Our (BVP) here, however, is time-dependent, strongly nonlinear and requires different analysis. In particular, an important tool for both, existence and higher regularity will be based on a result by Attouch and Damlamian [1] on abstract evolution equations associated

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with time-dependent subdifferentials. In order to obtain higher regularity we follow the same idea as in the linear case [14], i.e. we formally differentiate the (BVP) with respect to t, thus arriving at a linear t-dependent parabolic problem to which the result of Attouch and Damlamian is applicable. Further differentiations with respect to t will produce similar time-dependent evolution problems. Consequently, under appropriate regularity assumptions on the data and compatibility conditions, we can obtain any order of regularity with respect to t. Similarly, by differentiating the equation (E) with respect to x, we can reach any level of regularity with respect to the space variable.

Let us point out that the regularity question for the (BVP) is important as an intermediate step in developing an asymptotic analysis of the telegraph system with small specific distributed inductance and with algebraic-differential boundary conditions (see [6] for particular cases of the (BVP)). Note that the telegraph (BVP) with algebraic-differential boundary conditions has a concrete physical meaning (see, e.g., [11] and the references therein).

On the other hand, the multivalued terms appearing in (E) and (BC) may come from feedback distributed or boundary controls (see [12] for different examples).

2 Preliminaries

In this section we collect a few results which will be needed in what follows.

Theorem 2.1 ([9, Appendix].) *Let X be a reflexive real Banach space and let $u \in L^p(a,b;X)$ with $-\infty < a < b < \infty$ and $1 < p < \infty$. Then the following two conditions are equivalent:*

(I) $u \in W^{1,p}(a,b;X)$*;*

(II) $\int_a^{b-\delta} \|u(t+\delta) - u(t)\|_X^p \, dt \le C\delta^p$ *for all* $\delta \in (0, b-a)$*,*

where C is some positive constant.

For $p = 1$, the implication (I) $\Rightarrow$ (II) *is still valid. Moreover,* (II) *holds if u is of bounded variation on* $[a,b]$*, even if X is not reflexive.*

As usual, by L^p and $W^{k,p}$ we denote the classical function and Sobolev spaces, respectively.

Now, let V, H be two real Hilbert spaces with $V \subset H$ densely and continuously. If H is identified with its dual H', then one obtains the usual Gelfand triple: $V \subset H \subset V'$, algebraically and topologically. Let us recall the definition of the space $W(a,b)$ introduced in [15, Chapter1]:

$$W(a,b) \ := \ \left\{ u \in L^2(a,b;V) \, ; u' \in L^2(a,b;V') \right\} .$$

Theorem 2.2 *If $u \in W(a,b)$ then $t \mapsto \|u(t)\|_H^2$ is absolutely continuous on $[a,b]$ and*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 \ = \ \langle u'(t) \, , \, u(t)\rangle_{V'\times V} \quad \textit{for all} \quad t \in (a,b) \, .$$

Theorem 2.3 ([1].) *Let $A(t) = \partial\varphi(t,\cdot)$ for $0 \le t \le T$, where $\varphi(t,\cdot) : H \to (-\infty, +\infty]$ are proper, convex, lower semicontinuous functions and where H is a real Hilbert space. Assume further that there exist a nondecreasing, absolutely continuous function $\gamma : [0,T] \to \mathbb{R}$ and some positive constants C_1, C_2 such that*

$$\varphi(t,v) \ \le \ \varphi(s,v) + [\gamma(t) - \gamma(s)] \cdot \left[\varphi(s,v) + C_1\|v\|_H^2 + C_2\right] , \tag{2.1}$$

for all $v \in H$ and $0 \le s \le t \le T$. Then, for every $u_0 \in D(\varphi(0,\cdot))$ and $f \in L^2(0,T;H)$ there exists a unique solution $u \in W^{1,2}(0,T;H)$ of the equation $u'(t) + A(t)u(t) \ni f(t)$ for a. a. $t \in (0,T)$ with the initial condition $u(0) = u_0$. In addition, there exists a function $h \in L^1(0,T)$ such that

$$\varphi(t,u(t)) \ \le \ \varphi(s,u(s)) + \int_s^t h(\tau)\, d\tau \quad \textit{for all} \quad 0 \ \le \ s \ \le \ t \ \le \ T \, . \tag{2.2}$$

Remark 2.4 The condition (2.1) implies that $D(\varphi(0,\cdot)) \subset D(\varphi(s,\cdot)) \subset D(\varphi(t,\cdot))$ for all $0 \le s \le t \le T$. Actually, in this paper we shall have only cases in which $D(\varphi(t,\cdot))$ is independent of t.

3 Existence and uniqueness

If $s_1(t)$ is a constant function then the (BVP) can be expressed as a Cauchy problem for a quasi-autonomous evolution equation in the product space $H = L^2(0,1) \times \mathbb{R}$. Moreover, in this case, by redefining β_1, the constant s_1 can be set zero. For details concerning this case see [13]. If $s_1(t)$ is not a constant function, then we are led to a time-dependent abstract Cauchy problem in H. In case that g is a single-valued linear function, then $(\mathrm{BC})_1$ can be homogenized by a simple change of the unknown, i.e.,

$$\widetilde{y}(x,t) \; = \; y(x,t) + x(1-x)s_1(t)\,,$$

again obtaining a quasi-autonomous Cauchy problem. Note, however, that we have to pay some price for this transformation since $s_1(t)$ must be chosen as sufficiently regular.

In addition, when g is a nonlinear function, such a change of the unknown y would transform the equation (E) into a time-dependent evolution equation. Fortunately, we can avoid this change of the unknown, since Theorem 2.3 by Attouch and Damlamian applies directly to the original (BVP). So, we equip $H = L^2(0,1) \times \mathbb{R}$ with the scalar product

$$\langle (h_1, a_1)\,,\, (h_2, a_2) \rangle_H \; := \; \int_0^1 h_1(x) h_2(x)\, dx + a_1 \cdot a_2$$

and $\|\cdot\|_H = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. The following assumption will also be used throughout this paper:

(A1) The mappings $g : D(g) \subset \mathbb{R} \to \mathbb{R}\,,\ \beta_i : D(\beta_i) \subset \mathbb{R} \to \mathbb{R}\,,\ i = 1,2\,,$ are all maximal monotone.

For the theory of monotone operators see for instance in [8], [9], [16].

In fact, (A1) amounts to saying that there exist three proper, convex, lower semicontinuous functions, say $j, j_i : \mathbb{R} \to (-\infty, +\infty]$ such that $g = \partial j$ and $\beta_i = \partial j_i$, $i = 1,2$ (see, e.g., [16, pp. 42]), where ∂j, ∂j_i denote the subdifferentials of j, j_i which are unique up to additive constants.

An additional hypothesis that we are going to employ is

(A2) $D(j) \cap D(j_i) \neq \emptyset$ for $i = 1,2$, where $D(j)\,,\ D(j_i)$ are the effective domains of j, j_i.

Now let us associate with the (BVP) the following Cauchy problem in $H = L^2(0,1) \times \mathbb{R}$:

$$\frac{d}{dt}\begin{pmatrix} y(\cdot,t) \\ \zeta(t) \end{pmatrix} + A(t)\begin{pmatrix} y(\cdot,t) \\ \zeta(t) \end{pmatrix} \ni \begin{pmatrix} f(\cdot,t) \\ s_2(t) \end{pmatrix} \quad \text{for} \quad 0 \,<\, t \,<\, T\,, \tag{3.1}$$

$$\left.\begin{pmatrix} y(\cdot,t) \\ \zeta(t) \end{pmatrix}\right|_{t=0} \; = \; \begin{pmatrix} y_0 \\ \zeta_0 \end{pmatrix}, \tag{3.2}$$

where $A(t) : D(A(t)) \subset H \to H$ is defined by

$$A(t)\begin{pmatrix} h \\ a \end{pmatrix} \; = \; \begin{pmatrix} -h'' + g(h(\cdot)) \\ h'(1) + \beta_2(1) \end{pmatrix}$$

with the domain

$$\begin{aligned} D(A(t)) \; = \; & \big\{ (h,a)/h \in H^2(0,1)\,,\ g(h(\cdot)) \in L^2(0,1)\,,\ h(0) \in D(\beta_1)\,, \\ & -h'(0) + \beta_1(h(0)) \ni s_1(t)\,,\ a = h(1) \in D(\beta_2) \big\}\,. \end{aligned}$$

We denote by $H^k(0,1)$ the usual Sobolev space of order k. In order to see the relationship between the (BVP) and the problem (3.1), (3.2), let us define the functional $\psi(t,\cdot) : H \to (-\infty, +\infty]$ by

$$\psi(t,(h,a)) \; := \; \begin{cases} \displaystyle\int_0^1 \left\{ \frac{1}{2} h'(x)^2 + j(h(x)) \right\} dx + j_1(h(0)) - s_1(t)h(0) + j_2(a)\,, \\ \text{if } h \in H^1(0,1)\,, j(h) \in L^1(0,1)\,,\ h(0) \in D(j_1) \text{ and } a = h(1) \in D(j_2)\,, \\ +\infty\,, \quad \text{otherwise}\,. \end{cases}$$

Theorem 3.1 *Assume that the assumptions* (A1), (A2) *are satisfied. If* $(f, s_2) \in L^2(0,T;H)$, $s_1 \in W^{1,1}(0,T)$ *and* $(y_0, \zeta_0) \in D := D(\psi(0,\cdot))$ *then the Cauchy problem* (3.1), (3.2) *has a unique solution*

$$(y, \zeta) \in W^{1,2}(0,T;H) \quad \textit{with} \quad y_x \in L^\infty\big(0,T;L^2(0,1)\big)\,. \tag{3.3}$$

If, in addition, $(y_0, \zeta_0) = (y_0, y_0(1)) \in D(A(0))$*,* $s_1 \in W^{1,2}(0,T)$ *and* $(f, s_2) \in W^{1,1}(0,T;H)$ *then*

$$(y, \zeta) \in W^{1,\infty}(0,T;H) \quad \textit{and} \quad y \in W^{1,2}\big(0,T;H^1(0,1)\big)\,. \tag{3.4}$$

P r o o f. Clearly, the effective domain of $\psi(t,\cdot)$ is independent of t : $D(\psi(t,\cdot)) = D$ and D is not empty. Indeed, one can find a pair $(h, a) \in D$; for instance, choose $a \in D \cap D\big(j_2\big)$ and $h(x) = ax + a_1(1-x)$ with $a_1 \in D(j) \cap D\big(j_1\big)$ (see (A2)).

By standard arguments it follows that $\psi(t,\cdot)$ is convex and lower semicontinuous. Its subdifferential, $\partial\psi(t,\cdot)$ coincides with the operator $A(t)$ defined above for all $t \in [0,T]$. Let us now check whether the key condition (2.1) of Theorem 2.3 is satisfied under the assumptions (A1), (A2) and $s_1 \in W^{1,1}(0,T)$. For any $(h,a) = (h, h(1)) \in D$ and $0 \le s \le t \le T$ we have

$$\psi(t,(h,h(1))) - \psi(s,(h,h(1))) \;=\; -h(0)[s_1(t) - s_1(s)] \;\le\; |h(0)| \int_s^t |s_1'(\tau)|\, d\tau\,. \tag{3.5}$$

On the other hand, the properties of proper, convex, lower semicontinuous functions imply that

$$\psi(s,(h,h(1))) \;\ge\; \frac{1}{2}\, \|h'\|^2_{L^2(0,1)} - C_1\, \|h\|_{L^2(0,1)} - C_2\, |h(0)| - C_3\, |h(1)| - C_4\,, \tag{3.6}$$

where C_i , $i = \overline{1,4}$ are some positive constants.

Since

$$|h(0)| \;\le\; |h(1)| + \left| \int_0^1 h'(s)\, ds \right| \;\le\; |h(1)| + \|h'\|_{L^2(0,1)}\,,$$

the estimate (3.6) implies

$$|h(0)| \;\le\; \psi(s,(h,h(1))) + C_5\, \|(h,h(1))\|^2_H + C_6\,, \tag{3.7}$$

where C_5, C_6 are again some positive constants.

Now, by (3.5) and (3.7) it follows that condition (2.1) is fulfilled with

$$\gamma(t) \;=\; \int_0^t |s_1'(\tau)|\, d\tau\,.$$

Therefore, according to Theorem 2.3, under the first set of our assumptions, there exists a unique solution $(y,\zeta) \in W^{1,2}(0,T;H)$ of the problem (3.1), (3.2). In particular, $\zeta(t) = y(1,t)$ for a.a. $t \in (0,T)$. In addition, by (2.2) with $s = 0$ we have

$$\psi(t,(y(\cdot,t)\,,\,\zeta(t))) \;\le\; C_7\,.$$

The combination of this estimate with (3.6) yields $y_x \in L^\infty(0,T;L^2(0,1))$ which concludes the proof of the first part of Theorem 3.1. Assume now that $(y_0, \zeta_0) = (y_0, y_0(1)) \in D(A(0))$, $s_1 \in W^{1,2}(0,T)$ and $(f, s_2) \in W^{1,1}(0,T;H)$. In order to derive (3.4) we can use standard arguments of the theory of evolution equations. So, if we denote

$$z(t) \;:=\; \begin{pmatrix} y(\cdot,t) \\ y(1,t) \end{pmatrix}\,, \quad F(t) \;:=\; \begin{pmatrix} f(\cdot,t) \\ s_2(t) \end{pmatrix}\,,$$

then it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\, \|z(t+\delta) - z(t)\|^2_H + \|y_x(\cdot,t+\delta) - y_x(\cdot,t)\|^2_{L^2(0,1)} \\ &\quad\le\; \langle F(t+\delta) - F(t)\,,\, z(t+\delta) - z(t)\rangle_H + [s_1(t+\delta) - s_1(t)] \cdot [y(0,t+\delta) - y(0,t)] \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a.} \quad 0 \,<\, t \,<\, t+\delta \,<\, T\,, \end{aligned} \tag{3.8}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\delta}\|z(\delta)-z(0)\|_H^2+\|y_x(\delta)-y_{0,x}\|_{L^2(0,1)}^2\\&\quad\le\ \langle F(\delta)-A(0)z(0)\,,\ z(\delta)-z(0)\rangle_H+[s_1(\delta)-s_1(0)]\cdot[y(0,\delta)-y_0(0)]\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for a.a.}\quad\delta\in(0,T)\,.\end{aligned}\tag{3.9}$$

For any function $k\in H^1(0,1)$ we have

$$|k(0)|\ \le\ C_8\big(\|k\|_{L^2(0,1)}+\|k'\|_{L^2(0,1)}\big)\tag{3.10}$$

since $C[0,1]$ is continuously embedded into $H^1(0,1)$. By using (3.10) one can deduce the estimate

$$\begin{aligned}&[s_1(t+\delta)-s_1(t)]\cdot[y(0,t+\delta)-y(0,t)]\\&\quad\le\ C_8\,|s_1(t+\delta)-s_1(t)|\cdot\|y(\cdot,t+\delta)-y(\cdot,t)\|_{L^2(0,1)}+\frac{C_8^2}{2}\,|s_1(t+\delta)-s_1(t)|^2\\&\qquad+\ \frac{1}{2}\,\|y_x(\cdot,t+\delta)-y_x(\cdot,t)\|_{L^2(0,1)}^2\,.\end{aligned}$$

Therefore, in view of (3.8) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\,\|z(t+\delta)-z(t)\|_H^2+\frac{1}{2}\,\|y_x(\cdot,t+\delta)-y_x(\cdot,t)\|_{L^2(0,1)}^2\\&\quad\le\ \frac{C_8^2}{2}\,|s_1(t+\delta)-s_1(t)|^2+C_8\,|s_1(t+\delta)-s_1(t)|\cdot\|y(\cdot,t+\delta)-y(\cdot,t)\|_{L^2(0,1)}\\&\qquad+\|F(t+\delta)-F(t)\|_H\cdot\|z(t+\delta)-z(t)\|_H\,.\end{aligned}\tag{3.11}$$

Similarly, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\delta}\,\|z(\delta)-z(0)\|^2+\frac{1}{2}\,\|y_x(\cdot,\delta)-y_{0,x}\|_{L^2(0,1)}^2\\&\quad\le\ \frac{C_8^2}{2}\,|s_1(\delta)-s_1(0)|^2+C_8\,|s_1(\delta)-s_1(0)|\cdot\|y(\cdot,\delta)-y(0)\|_{L^2(0,1)}\\&\qquad+\ \|F(\delta)-A(0)z(0)\|_H\cdot\|z(\delta)-z(0)\|_H\,.\end{aligned}\tag{3.12}$$

Integrating (3.11), (3.12) over $[0,t]$, $[0,\delta]$, respectively, we get

$$\begin{aligned}&\frac{1}{2}\,\|z(t+\delta)-z(t)\|_H^2+\frac{1}{2}\int_0^t\|y_x(\cdot,\tau+\delta)-y_x(\cdot,\tau)\|_{L^2(0,1)}^2\,d\tau\\&\quad\le\ \frac{1}{2}\,\|z(\delta)-z(0)\|_H^2+\frac{C_8^2}{2}\int_0^t|s_1(\tau+\delta)-s_1(\tau)|^2\,d\tau\\&\qquad+\ \int_0^t(\|F(\tau+\delta)-F(\tau)\|_H+C_8\,|s_1(\tau+\delta)-s_1(\tau)|)\cdot\|z(\tau+\delta)-z(\tau)\|_H\,d\tau\,,\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}&\frac{1}{2}\,\|z(\delta)-z(0)\|^2+\frac{1}{2}\int_0^\delta\|y_x(\cdot,\tau)-y_{0,x}\|_{L^2(0,1)}^2\,d\tau\\&\quad\le\ \frac{C_8^2}{2}\int_0^\delta|s_1(\tau)-s_1(0)|^2d\tau+\int_0^\delta(\|F(\tau)-A(0)z(0)\|_H+C_8\,|s_1(\tau)-s_1(0)|)\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\|z(\tau)-z(0)\|_H\,d\tau\,.\end{aligned}\tag{3.14}$$

Now taking into account the estimate

$$|s_1(\tau)-s_1(0)|^2\ \le\ \tau\int_0^\tau s_1'(\theta)^2\,d\tau\ \le\ C_9\tau$$

we obtain from (3.14) the inequality

$$\frac{1}{2}\|z(\delta)-z(0)\|_H^2 \le \frac{C_8^2C_9}{4}\delta^2+\int_0^\delta(\|F(\tau)-A(0)z(0)\|_H+C_8|s_1(\tau)-s_1(0)|)\times\|z(\tau)-z(0)\|_H\,d\tau\,,$$

which implies by a Gronwall's type lemma (see, e.g., [16, pp. 47] which still works for our slightly different case) that

$$\|z(\delta)-z(0)\|_H \le C_{10}\delta+\int_0^\delta(\|F(\tau)-A(0)z(0)\|_H\,d\tau+C_8\int_0^\delta|s_1(\tau)-s_1(0)|\,d\tau\,. \tag{3.15}$$

On the other hand, employing the same lemma we can derive from (3.13) the estimate

$$\begin{aligned}\|z(t+\delta)-z(t)\|_H \le&\left(\|z(\delta)-z(0)\|^2+C_8^2\int_0^t|s_1(\tau+\delta)-s_1(\tau)|^2\,d\tau\right)^{\frac{1}{2}}\\&+\int_0^t(\|F(\tau+\delta)-F(\tau)\|_H+C_8|s_1(\tau+\delta)-s_1(\tau)|)\,d\tau\,.\end{aligned}\tag{3.16}$$

Now, it follows from (3.15) and (3.16) that

$$\begin{aligned}\|z(t+\delta)-z(t)\|_H \le C_{11}\Bigg\{&\delta+\int_0^\delta(\|F(\tau)-A(0)z(0)\|_H\,d\tau+\int_0^\delta|s_1(\tau)-s_1(0)|\,d\tau\\&+\left(\int_0^t|s_1(\tau+\delta)-s_1(\tau)|^2\,d\tau\right)^{\frac{1}{2}}+\int_0^t\|F(\tau+\delta)-F(\tau)\|_H\,d\tau\\&+\int_0^t|s_1(\tau+\delta)-s_1(\tau)|\,d\tau\Bigg\}\,.\end{aligned}\tag{3.17}$$

In view of our assumptions on $s_1,\ s_2$ and f, we can conclude by (3.17) that $z_t\in L^\infty(0,T;H)$. It remains to show that $y_x\in W^{1,2}\big(0,T;L^2(0,1)\big)$. To this end, let us come back to (3.13). Recalling that

$$\|z(t+\delta)-z(t)\|_H = \left\|\int_t^{t+\delta}z'(\theta)\,d\theta\right\|_H \le C_{12}\delta$$

for all $0\le t\le t+\delta\le T$ and taking into account Theorem 2.1, we can see by (3.13) that

$$\int_0^{T-\delta}\|y_x(\cdot,\tau+\delta)-y_x(\cdot,\tau)\|_H^2\,d\tau \le C_{13}\delta^2\,. \tag{3.18}$$

We have neglegted the first term of the left-hand side in (3.13) and used our assumptions on $s_1,\ s_2$ and f. Thus, again according to Theorem 2.1, (3.18) implies that $y_x\in W^{1,2}\big(0,T;L^2(0,1)\big)$ as asserted. The proof of Theorem 3.1 is now complete. □

Remark 3.2 The solution $(y(\cdot,t),\zeta(t))$ given by Theorem 2.1 is a solution of the (BVP), as can be checked by using the fact that $\zeta(t)=y(1,t)$ for a.a. $t\in(0,T)$. If we are in the last case of Theorem 3.1 we have even more: $\zeta(t)=y(1,t)$ for all $t\in[0,T]$ since $y\in C\big(\overline{D}_T\big)$. Let us point out that the condition $\zeta_0=y_0(1)$ employed in Theorem 3.1 is a compatibility condition. But the question of existence for the (BVP) can also be discussed beyond this compatibility condition. Actually, Theorem 2.3 still works for $u_0\in\overline{D(\varphi(0,\cdot))}$ and $f\in L^2(0,T;H)$ but with a weaker conclusion, $\sqrt{t}u'\in L^2(0,T;H)$ instead of $u'\in L^2(0,T;H)$ which also applies to our (BVP).

But we can do even more, namely we can assume that $(y_0,\zeta_0)\in\overline{D}$, $(f,s_2)\in L^1(0,T;H)$, $s_1\in L^2(0,T)$ and derive the existence and uniqueness of a weak solution $(y,\zeta)\in C([0,T];H)$ with $y_x\in L^2(D_T)$ of the associated Cauchy problem (3.1), (3.2). More precisely, (y,ζ) is obtained as a limit in $C([0,T];H)$ of solutions given by Theorem 3.1 corresponding to smooth data that approximate (y_0,ζ_0,f,s_1,s_2).

In particular, (y_0, ζ_0) can be approximated by elements of D. This (y, ζ) can be considered as a generalized solution of the (BVP). The fact that $(y_0, \zeta_0) \in \overline{D}$ means that $\zeta_0 \in \overline{D(j_2)} = \overline{D(\beta_2)}$ and $y_0 \in L^2(0,1)$ with $y_0(x) \in \overline{D(j)} = \overline{D(g)}$ for a.a. $x \in (0,1)$. Hence, now the compatibility condition $\zeta_0 = y_0(1)$ is meaningless. Note also that ζ_0 is not present in the (BVP) and so we have nonuniqueness for the (BVP) since ζ_0 can be chosen arbitrarily in $\overline{D(j_2)}$. For a detailed analysis of this matter the reader is referred to [13]. Here we are mainly interested in higher regularity of the solution which requires smooth y_0 as well as the compatibility condition $y_0(1) = \zeta_0$.

Remark 3.3 Assume that the second set of the hypotheses of Theorem 2.1 is fulfilled. If the set $B := \{y(x,t); (x,t) \in \overline{D}_T\}$ would be included in $\operatorname{Int} D(g)$ then $\bigcup_{(x,t)\in\overline{D}_T} g(x,t)$ is bounded (cf. Rockafellar's Theorem; see, e.g. [16, Theorem 1.1 and Remark 1.1, pp. 18-19]). Therefore we get from (E) the additional regularity property $y \in L^\infty\big(0,T; H^2(0,1)\big)$. For example, this is the case when g is everywhere defined on $\mathbb{R}$.

It seems that for multivalued g, β_1, β_2 the above level of regularity cannot be exceeded. But the regularity can be improved if g, β_1, β_2 are single-valued and smooth. This matter is discussed in the next section.

4 Higher regularity of the solution

In this section we assume that g, β_1, β_2 are all single-valued and, hence, we shall understand that any inclusion relation involving these applications is actually an equality. The main result here is the following:

Theorem 4.1 *Assume that*

$$f \in W^{1,2}\big(0,T; L^2(0,1)\big)\,, \quad s_1 \in W^{2,1}(0,T)\,, \quad s_2 \in W^{1,2}(0,T)\,; \tag{4.1}$$

g, β_1, β_2 are all everywhere defined on $\mathbb{R}$, single-valued and belong to $W^{2,\infty}_{\text{loc}}(\mathbb{R})$ satisfying

$$g' \ \ge\ 0\,,\ \beta_1' \ \ge\ 0\,,\ \beta_2' \ \ge\ 0 \ \textit{ on } \mathbb{R}\,; \tag{4.2}$$

$$(y_0, y_0(1)) \in D(A(0))\,; \tag{4.3}$$

$$\begin{pmatrix} f(\cdot,0) \\ s_2(0) \end{pmatrix} - A(0)\begin{pmatrix} y_0 \\ y_0(1) \end{pmatrix} \in V\,, \tag{4.4}$$

where V is the space of all pairs $(\varphi, \varphi(1))$ with $\varphi \in H^1(0,1)$.

Then the solution y of the (BVP) belongs to $W^{2,2}\big(0,T; L^2(0,1)\big) \cap W^{1,\infty}\big(0,T; H^1(0,1)\big) \cap C\big([0,T]; H^2(0,1)\big)$ and $y(1,\cdot) \in W^{2,2}(0,T)$.

Proof. According to Theorem 3.1 and Remark 3.1, the (BVP) has a unique solution y satisfying (3.4). Clearly, the space V introduced above is a real Hilbert space whose scalar product is the sum of the usual scalar product of $H^1(0,1)$ and the ordinary multiplication in $\mathbb{R}$. Denote by V' its dual, by $(\cdot,\cdot)_{V'V}$ the duality pairing between V and V' and by $(\cdot,\cdot)$ the L^2-duality paring between $H^1(0,1)$ and $\big(H^1(0,1)\big)'$. Then we have from Equation (3.1)

$$\begin{aligned}
&\left(\frac{d}{dt}\begin{pmatrix} y(\cdot,t) \\ y(1,t) \end{pmatrix},\ \begin{pmatrix} \varphi \\ \varphi(1) \end{pmatrix}\right)_{V'V} + (y_x(\cdot,t), \varphi') + (g(y(\cdot,t)), \varphi) + \beta_1(y(0,t))\varphi(0) \\
&- \ s_1(t)\varphi(0) + \beta_2(y(1,t))\varphi(1) \\
&\quad = \ (f(\cdot,t), \varphi) + s_2(t)\varphi(1) \quad \text{for all } \ \varphi \in H^1(0,1) \ \text{ and a.a. } \ t \in (0,T)\,.
\end{aligned} \tag{4.5}$$

Denote as before

$$z(t) \ := \ \begin{pmatrix} y(\cdot,t) \\ y(1,t) \end{pmatrix}\,.$$

We are going to show that $z' = dz/dt \in W^{1,2}(0,T;V')$. In view of Theorem 2.1, it is sufficient to show that for every $\delta \in (0,T)$

$$\int_0^{T-\delta} \|z'(t+\delta) - z'(t)\|^2_{V'}\, dt \ \le\ C\delta^2\,, \tag{4.6}$$

where C is a positive constant. Indeed, taking into account (3.4) and (4.2), which in particular implies that g, β_1, β_2 are all Lipshitzian on bounded intervals, one obtains from (4.5) the estimate

$$\begin{aligned} \|z'(t+\delta) - z'(t)\|^2_{V'} \;\le\; \widetilde{C}\Big\{&\|y(\cdot,t+\delta) - y(\cdot,t)\|^2_{H^1(0,1)} + \|f(\cdot,t+\delta) - f(\cdot,t)\|^2_{L^2(0,1)} \\ &+ |s_1(t+\delta) - s_1(t)|^2 + |s_2(t+\delta) - s_2(t)|^2\Big\} \end{aligned} \tag{4.7}$$

for a.a. $t \in (0, T-\delta)$, where $\widetilde{C}$ is a positive constant. By (3.4), (4.1) and (4.7) one obtains the desired estimate (4.6) (cf. Theorem 2.1). Therefore we can differentiate (4.5) to find

$$\begin{aligned} &\left(\begin{pmatrix} v_t(\cdot,t) \\ v_t(1,t)\end{pmatrix}, \begin{pmatrix} \varphi \\ \varphi(1)\end{pmatrix}\right)_{V'V} + (v_x(\cdot,t),\varphi') + (\alpha(\cdot,t)z(\cdot,t),\varphi) + \alpha_1(t)z(0,t)\varphi(0) + \alpha_2(t)z(1,t)\varphi(1) \\ &= s_1'(t)\varphi(0) + s_2'(t)\varphi(1) + (f_t(\cdot,t),\varphi) \;\text{ for all }\; \varphi \in H^1(0,1) \;\text{ and a.a. }\; t \in (0,T)\,, \end{aligned} \tag{4.8}$$

where

$$v \;:=\; y_t\,, \quad \alpha(\cdot,t) \;:=\; g'(y(\cdot,t))\,, \quad \alpha_1(t) \;:=\; \beta_1'(y(0,t))\,, \quad \alpha_2(t) \;:=\; \beta_2'(y(0,t))\,.$$

We also have

$$\begin{pmatrix} v(\cdot,0) \\ v(1,0)\end{pmatrix} \;=\; \begin{pmatrix} f(\cdot,0) \\ s_2(0)\end{pmatrix} - A(0)\begin{pmatrix} y_0 \\ y_0(1)\end{pmatrix}. \tag{4.9}$$

Clearly, the problem (4.8), (4.9) has a unique solution in $W^{1,2}(0,t;V') \cap L^2(0,T;V)$. Indeed, if the right-hand sides of (4.8) and (4.9) are set equal to zero and we take in (4.8) $\varphi = v(\cdot,t)$, then we obtain (cf. Theorem 2.2 and (4.2))

$$\frac{d}{dt}\Big(\|v(\cdot,t)\|^2_{L^2(0,1)} + v(1,t)^2\Big) \;\le\; 0 \quad \text{for a.a.} \quad t \in (0,T)\,,$$

which implies $v \equiv 0$ and, hence, uniqueness is proved.

Formally, $(v(\cdot,t)\,, v(1,t))$ satisfies the following Cauchy problem in H:

$$\frac{d}{dt}\begin{pmatrix} v(\cdot,t) \\ v(1,t)\end{pmatrix} + A_1(t)\begin{pmatrix} v(\cdot,t) \\ v(1,t)\end{pmatrix} \;=\; \begin{pmatrix} f_t(\cdot,t) \\ s_2'(t)\end{pmatrix} \quad \text{for} \quad 0 \,<\, t \,<\, T\,, \tag{4.10}$$

$$\begin{pmatrix} v(\cdot,0) \\ v(1,0)\end{pmatrix} \;=\; \begin{pmatrix} f(\cdot,0) \\ s_2(0)\end{pmatrix} - A(0)\begin{pmatrix} y_0 \\ y_0(1)\end{pmatrix}, \tag{4.11}$$

where $A_1(t) : D(A_1(t)) \subset H \to H$ is defined by

$$A_1(t)\begin{pmatrix} h \\ a\end{pmatrix} \;=\; \begin{pmatrix} -h''(\cdot) + \alpha(\cdot,t)h(\cdot) \\ h'(1) + \alpha_2(t)a\end{pmatrix}$$

with

$$D(A_1(t)) \;=\; \big\{(h,a);\, h \in H^2(0,1)\,,\; a = h(1)\,,\; -h'(0) + \alpha_1(t)h(0) = s_1'(t)\big\}\,.$$

An easy computation shows that $A_1(t)$ is the subdifferential of the function $\psi_1(t,\cdot) : H \to (-\infty,\infty]$ defined by

$$\psi_1\Big(t, \begin{pmatrix} h \\ a\end{pmatrix}\Big) \;=\; \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 \big\{h'(x)^2 + \alpha(x,t)h(x)^2\big\}\,dx + \dfrac{1}{2}\,\alpha_1(t)h(0)^2 - s_1'(t)h(0) + \dfrac{1}{2}\,\alpha_2(t)a^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } h \in H^1(0,1) \text{ and } a \,=\, h(1)\,, \\ +\infty \quad \text{otherwise}\,. \end{cases}$$

The set $D(\psi_1(t,\cdot)) = V$ and $\psi_1(t,\cdot)$ is convex and lower semicontinuous on H. Therefore, $A_1(t) = \partial\psi_1(t,\cdot)$ is a maximal monotone operator. In order to apply Theorem 2.3 to the problem (4.10), (4.11), we have to check the key condition (2.1). To this end, let $(h,a) = (h,h(1)) \in V$ and let $0 \le s \le t \le T$.

We have

$$
\begin{aligned}
&\psi_1(t,(h,h(1))) - \psi_1(s,(h,h(1))) \\
&\quad = \frac{1}{2}\int_0^1 [\alpha(x,t)-\alpha(x,s)]h(x)^2\,dx + \frac{1}{2}[\alpha_1(t)-\alpha_1(s)]h(0)^2 - [s_1'(t)-s_1'(s)]h(0) \\
&\quad + \frac{1}{2}[\alpha_2(t)-\alpha_2(s)]h(1)^2 \\
&\quad \le \int_s^t \|\alpha_\tau(\cdot,\tau)\|_{L^\infty(0,1)}\,d\tau\cdot\frac{1}{2}\|h\|^2_{L^2(0,1)} + \int_s^t |\alpha_1'(\tau)|\,d\tau\cdot\frac{1}{2}h(0)^2 \\
&\quad + \int_s^t |s_1''(\tau)|\,d\tau\cdot|h(0)| + \int_s^t |\alpha_2'(\tau)|\,d\tau\cdot\frac{1}{2}h(1)^2\,.
\end{aligned} \tag{4.12}
$$

On the other hand,

$$
\psi_1(s,(h,h(1))) \;\ge\; -s_1'(s)h(0)\,. \tag{4.13}
$$

By (4.12) and (4.13) we can see that (2.1) is fulfilled with

$$
\begin{aligned}
\gamma(t) &= \int_0^t \big\{\|\alpha_\tau(\cdot,\tau)\|_{L^\infty(0,1)} + |\alpha_1'(\tau)| + |\alpha_2'(\tau)| + |s_1''(\tau)|\big\}\,d\tau \\
&= \int_0^t \big\{\|g''(y(\cdot,\tau))\cdot y_\tau(\cdot,\tau)\|_{L^\infty(0,1)} + |\beta_1''(y(0,\tau))\cdot y_\tau(0,\tau)| + |\beta_2''(y(1,\tau))\cdot y_\tau(1,\tau)| \\
&\qquad + |s_1''(\tau)|\big\}\,d\tau\,.
\end{aligned}
$$

Therefore, Theorem 2.3 is again applicable and, hence, the problem (4.10), (4.11) has a unique solution $(v(\cdot,t),v(1,t)) \in W^{1,2}(0,T;H)$ with $v_x \in L^\infty(0,T;L^2(0,1))$ (cf. (4.13) and (2.2)). Clearly, this is also a solution of (4.8), (4.9) and so $v = y_t$, which implies that

$$
y \in W^{2,2}\big(0,T;L^2(0,1)\big)\cap W^{1,\infty}\big(0,T;H^1(0,1)\big)\,,\quad y(1,\cdot)\in W^{2,2}(0,T)\,.
$$

Moreover, by using the equation (E) we can see that $y \in C\big([0,T];H^2(0,1)\big)$. The proof is now complete. □

Remark 4.2 The condition $(y_0,\zeta_0) = (y_0,y_0(1)) \in D(A(0))$ which was employed in Theroems 3.1 and 4.1 expresses a certain regularity of y_0 and some compatibility with the boundary conditions (BC):

$$
\zeta_0 \;=\; y_0(1)\in D(\beta_2)\,,\quad y_0(0)\in D(\beta_1)\quad\text{and}\quad -\,y_0'(0)+\beta_1(y_0(0)) \ni s_1(0)\ \text{or}\ =\ s_1(0)\,.
$$

In order to have higher regularity of the solution, we also require in Theorem 4.1 the condition (4.4) which in particular implies the following additional compatibility condition:

$$
f(1,0) + y_0''(1) - g(y_0(1)) \;=\; s_2(0) - y_0'(1) - \beta_2(y_0(1))\,.
$$

Remark 4.3 If in Theorem 4.1 we assume more regularity of f with respect to the space variable, then higher regularity of y with respect to x can also be derived by using the equation (E) and the conclusion of Theorem 4.1. For instance, the equations

$$
\begin{aligned}
y_{xxx} &= y_{xt} + g'(y)y_x - f_x\,,\\
y_{xxxx} &= y_{txx} + g''(y)y_x^2 + g'(y)y_{xx} - f_{xx}\\
&= y_{tt} + g'(y)y_t - f_t + g''(y)y_x^2 + g'(y)y_{xx} - f_{xx}\,,
\end{aligned}
$$

imply higher regularity with respect to x.

On the other hand, we can continue the above procedure to increase the regularity with respect to t under additional assumptions concerning regularity of the data and their compatibility with the boundary conditions (BC). Note that further differentiations with respect to t of the boundary value problem (BVP) lead to linear t-dependent boundary value problems of type (4.8), (4.9) (or (4.10), (4.11)). Therefore, the $H^k(D_T)$-regularity can be attained for every k and, hence, the $C^k(\overline{D}_T)$-regularity can also be accomplished as well for every k, due to the classical Sobolev embedding theorem.

Remark 4.4 We can investigate by the same technique a slightly modified (BVP), where $y_t(x,t)$ and $y_t(1,t)$ are multiplied by some positive constants c and c_1, respectively. Then the modification of the scalar product in $H = L^2(0,1) \times \mathbb{R}$ in the form

$$\langle (h_1, a_1), (h_2, a_2) \rangle_{H,c,c_1} \;=\; c \int_0^1 h_1(x) h_2(x)\, dx + c_1 a_1 a_2 \,,$$

will imply the monotonicity of the operators $A(t)$ and $A_1(t)$ involved in our treatment.

Remark 4.5 As was already mentioned in the introduction, the case of algebraic-algebraic boundary conditions was investigated in the paper [7]. We can also study the case of differential-differential boundary conditions, i.e. the situation when both the boundary conditions are of type $(\mathrm{BC})_2$. An appropriate framework for this case is the product space $H_1 = L^2(0,1) \times \mathbb{R}^2$; and the corresponding unknown function is $t \mapsto (y(\cdot,t), y(0,t), y(1,t))$. Then, with the same techniques as in this paper, corresponding results can be obtained.

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