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Closed range mild solution operators and nonconvex optimal control via orthogonality and generalized gradients☆

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1. Introduction

Let X be a Banach space, X^* its dual and H a Hilbert space. Denote by $\langle \cdot,\cdot \rangle$ the duality pairing (i.e. $x^*(x) = \langle x^*, x\rangle$) and by $\|\cdot\|$ the norm on both X and X^*.

For simplicity of writing, the same notations will be used to denote the inner product and norm of both H and $L^2(0,T;X)$. Consider the T-periodic problem in X (with $T > 0$):

$$y'(t) = A(t)y(t) + f(t), \quad \text{a.e. } t \in (0,T), \quad y(0) = y(T), \tag{1.1}$$

where $A(t)$: $D(A(t)) = D \subset X \to X$ are linear densely defined operators (with time-independent domain D) generating an evolution operator $U(t,s)$ (in the usual sense, see e.g. [16], p. 23, Definition 3.3) on the Banach space X, $f \in L^2(0,T;X)$ and $T > 0$.

Definition 1.1. By a strong solution of the periodic problem (1.1) we mean a function $y \in W^{1,2}(0,T;X)$ satisfying (1.1) almost everywhere on $(0,T)$ (in short a.e. $t \in (0,T)$).

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Definition 1.2. By a mild solution u of (1.1) we mean a function $u \in C([0,T];H)$ given by the variation of constants formula:

$$y(t) = U(t,0)y(0) + \int_0^t U(t,s)f(s)\,\mathrm{d}s, \quad y(T) = y(0), \quad \forall t \in [0,T]. \tag{1.2}$$

Definition 1.3. A function $y \in L^2(0,T;H)$ is said to be a weak solution to the periodic problem (1.1) if

$$\int_0^T \langle y(t), \varphi'(t) + A^*(t)\varphi(t)\rangle\,\mathrm{d}t = -\int_0^T \langle f(s), \varphi(s)\rangle\,\mathrm{d}s \tag{1.3}$$

$\forall \varphi \in W^{1,2}_{T,A^*}(0,T;H)$, with

$$W^{1,2}_{T,A^*}(0,T;H) = \{\varphi \in W^{1,2}(0,T;H);\ \varphi(0) = \varphi(T), \varphi(t) \in D(A^*(t)),$$
$$\text{a.e. } t \in (0,T), t \to A^*(t)\varphi(t) \text{ is in } L^2(0,T;H)\}, \tag{1.4}$$

where $A^*(t)$ is the adjoint of $A(t)$ and $D(A(t)) = D$-independent of t.

This space is dense in $L^2(0,T;H)$ (see (2.8)). Obviously, the following space (which is also dense in $L^2(0,T;H)$) will be needed, too

$$W^{1,2}_{T,A}(0,T;H) = \{\psi \in W^{1,2}(0,T;H);\ \psi(0) = \psi(T), \psi(t) \in D(A(t)),$$
$$\text{a.e. } t \in (0,T),\ t \to A(t)\psi(t) \text{ is in } L^2(0,T;H)\}. \tag{1.5}$$

The mild solution of problem

$$y' = A(t)y + f(t), \quad y(T) = ay(0), \quad a \in R,\ a \neq 0 \tag{1.6}$$

is given by

$$y(t) = U(t,0)y(0) + \int_0^t U(t,s)f(s)\,\mathrm{d}s, \quad y(T) = ay(0), \quad \forall t \in [0,T]. \tag{1.7}$$

Define *the mild solution operator* $M_a : C([0,T];H) \to L^2(0,T;H)$ by

$$M_a y = f, \quad \text{iff (1.7) holds.} \tag{1.8}$$

This means that the domain $D(M_a)$ of M_a is the following one:

$$D(M_a) = \{y \in C([0,T];H),\ y(T) = ay(0); \exists f \in L^2(0,T;H),\ (1.7) \text{ holds}\}. \tag{1.9}$$

For simplicity, set $M_1 = M$. Therefore

$$D(M) = \{y \in C([0,T];H), y(T) = y(0);$$
$$\exists f \in L^2(0,T;H),\ (1.7) \text{ holds with } a = 1\}. \tag{1.9$'$}$$

In Section 2, the main properties of M are presented. These properties extend some recent results (from the autonomous case) of Barbu [3].

Section 3 is mainly concerned with the following optimal control problem

$$(P)\quad \begin{aligned} &\text{(Locally) Minimize } L(y,u) \\ &\text{subject to } y' = A(t)y + B(u) + f, \quad y(T) = ay(0),\ a = \pm 1, \end{aligned}$$

where $A(t) : D(A(t) \subset X \to X$ are unbounded closed linear operators with dense domain $D(A(t)) = D$-independent of t and closed range $R(A(t))$, $B : U \to X$ is a (Fréchet) differentiable map on an open subset U of X and L is a local Lipschitz functional.

If solution y (corresponding to a given $Bu + f$) is unique (i.e. if the null space $N(M)$ is zero), we say that the system above is in *the non-resonant case*. Otherwise, the system is in *the resonant case*. The resonant case is more important from both practical and mathematical point of view.

Let us recall that every $f \in L^2(0,T;H)$ has a Fourier representation

$$f = \sum_{m\in Z} f_m e^{\mu_m it} \quad \text{with } f_m = T^{-1}\int_0^T f(t)e^{-\mu_m it}\,dt, \quad m \in Z,$$
$$\text{and } \mu_m = 2m\pi T^{-1}, \quad \|f\|^2 = \sum_{m\in Z} |f_m|^2, \tag{1.10}$$

where $\|f\|$ denotes the norm of f in $L^2(0,T;H)$ and $|f_m|$ the modulus of the Fourier coefficients f_m of f.

Let $f \in R(M)$. This means that there is $y \in C([0,T];H)$

$$y = \sum_{m\in Z} y_m e^{\mu_m it} \quad \text{with } y_m = T^{-1}\int_0^T y(t)e^{-\mu_m it}\,dt, \quad m \in Z \tag{1.11}$$

with $\|y\|^2 = \sum_{m\in Z} |y_m|^2$, satisfying (1.2). The above series are convergent in the strong sense of $L^2(0,T;H)$.

Remark 1.1. In our case here, the T-anti-periodic condition $y(0) = -y(T)$ can be reduced to the $2T$-periodic condition $y(0) = y(2T)$ with $L^2_{\text{anp}}(0,2T;H)$ in place of $L^2(0,T;H)$, where

$$L^2_{\text{anp}}(0,2T;H) = \{f \in L^2(0,2T;H);\ f(t+T) = -f(t),\ \text{a.e. } t \in [0,T]\} \tag{1.12}$$

the space of all T-anti-periodic functions. More precisely, all the results in this paper can be restated (are valid) for the T-anti-periodic condition, with $L^2_{\text{anp}}(0,2T;H)$ in place of $L^2(0,T;H)$. This is not always the case. For example, the results in [2] on T-anti-periodic solutions cannot be derived from results on $2T$-periodic solutions. The key fact here is that every (T-anti-periodic) function f in $L^2_{\text{anp}}(0,2T;H)$ has the following Fourier representation:

$$f = \sum_p f_p e^{v_p it} \quad \text{with } f_p = (2T)^{-1}\int_0^{2T} f(t)e^{-v_p it}\,dt, \quad p = \pm 1, \pm 3, \pm 5, \ldots$$
$$\text{and } v_p = p\pi T^{-1}, \quad \|f\|^2 = \sum_p |f_p|^2. \tag{1.13}$$

Clearly, a strong solution of (1.1) is a weak solution. This follows multiplying (1.1) by φ and integrating by parts. The converse statement may not be true. However, a weak solution $y \in W^{1,2}(0,T;H)$ of (1.1) is a strong solution. This is less obvious and it follows from (1.2) integrating by parts and using density arguments. This is however routine, so it is omitted. A strong solution is a mild solution, too. In the infinite dimensional case (i.e. $\dim X = \infty$) there are nondifferentiable mild solutions (and obviously, such mild solutions are not strong solutions). A differentiable mild solution is a strong solution [16, p. 35, or [17], Section 5.6]. In the finite dimensional case, these two notions are equivalent. Moreover, it was recently proved (Barbu [3]) that in Hilbert spaces the notions of weak and mild solutions coincide in the autonomous case $A(t) = A$-independent of t. In the nonautonomous case here, this is not known. However, a mild solution y with $y(0) = y(T)$ which is a strong limit in $C([0,T];H)$ of strong solutions y_n (even with $y_n(0) \neq y_n(T)$) is a weak solution, too. This follows integrating by parts and taking into account that $\lim y_n(0) = y(0) = y(T) = \lim y_n(T)$. The motivation of this paper is the following:

1. It extends some recent results of Barbu [3] to the nonautonomous case, which will lead to new applications (time-dependent constraints).

2. Some of our results here such as: the compactness of $\tilde{M}$ (Proposition 2.2), Proposition 2.1, the main results of Section 3, are new even in the autonomous case $A(t) = A$-independent of t.

3. It points out new open problems such as: the problem of weak solutions in the nonautonomous case above, the problem of the extension of the main results of this paper to spaces more general than the Hilbert spaces here, the density of $D(M_a)$ in $L^2(0,T;H)$ (see Remark 2.1).

The technique we are using here is based on the following facts:

(a) The tangent cone to the constraints set $C = \{(y,u); My = Bu + f\}$ is a linear space (in all of our cases here).

(b) The orthogonality between the null space $N(M)$ of a closed range densely defined and closed operator M and the range $R(M^*)$ of its dual M^* (which holds in any Banach space X).

(c) The "Range Relationship": "either $R(M) \subset R(B'(u))$ or $R(B'(u)) \subset R(M)$".

(d) A local Lipschitz function L is locally everywhere differentiable in the generalized sense of Clarke [9] (i.e. it admits generalized directional derivatives and generalized gradients).

Among related recent papers on the theory of optimization and optimal control of the systems in the resonant case, we mention:

(I) Barbu and Pavel [4–6], and Barbu [3] under (b) and convex cost functionals L.

(II) Pavel and his collaborators [10–14], under (a), (b), (c) and convex L.

(III) Aizicovici–Motreanu and Pavel [1], under (a)–(d), B possible nonlinear, and nonconvex cost functionals L.

2. Main results

In the autonomous case $A(t) = A$ independent of t, the infinitesimal generator of a strongly continuous semigroup $S(t) = e^{tA}$, we have $U(t,s) = e^{(t-s)A}$, so the mild solution of the problem

$$y' = Ay + f(t), \quad y(0) = y(T) \tag{2.1}$$

is given by

$$y(t) = e^{tA} y(0) + \int_0^t e^{(t-s)A} f(s)\, ds, \quad y(0) = y(T), \quad \forall t \in [0,T]. \tag{2.2}$$

A simple classical result on (2.1) in the finite dimensional case ($\dim H < \infty$) asserts that (2.1) has a unique solution, iff the homogeneous equation $u' = Au, u(0) = u(T)$ admits only the trivial solution, which in turn is equivalent to: $\det(I - e^{TA}) \neq 0$, or $1 \in \rho(e^{TA})$. Its extension to infinite dimensional case is the following one.

Theorem 2.1 (Prüss [18]). *The following three conditions are equivalent*:

(1) *For every continuous function* $f \in C([0,T];H)$, *the problem* (2.1) *has a unique mild solution.*

(2) $1 \in \rho(e^{TA})$.

(3) $\sup_{m \in Z} \|(\mu_m i - A)^{-1}\| < \infty$.

The case $R(M)$-closed is important in control problems and interesting in itself, too. The result is the following (due to Barbu [3]):

$$R(M) \text{ is closed in } L^2(0,T;H) \text{ iff } R(I - e^{TA}) \text{ closed in } H. \tag{2.3}$$

Recall that M (or its graph $G(M)$) is said to be strongly–weakly closed in $L^2(0,T;H)$ if $My_n = f_n$ with $y_n \to y$ and $f_n \rightharpoonup f$ (weakly) in $L^2(0,T;H)$, imply $My = f$.

The basic properties of M_a (with $a \neq 0$) are given by

Theorem 2.2. 1. $R(M_a) = L^2(0,T;X)$ *iff* $R(aI - U(T,0)) = X$.

2. $R(M_a) = L^2(0,T;X)$ *and the uniqueness of the solution* y *to* $M_a y = f$ *is equivalent to* $a \in \rho(U(T,0))$. *In this case we have* (*for some* $c > 0$)

$$\|M_a^{-1} f\|_{C([0,T];X)} \leqslant c\|f\|_{L^1(0,T;X)}, \quad \forall f \in L^2(0,T;X). \tag{2.4}$$

3. $R(M_a)$ *is closed in* $L^2(0,T;X)$ *iff* $R(aI - U(T,0))$ *is closed in* X.

4. *If* $U(T,0)$ *is compact*, *then* $R(M_a)$ *is closed in* $L^2(0,T;X)$ *and the null space* $N(M_a)$ *of* M_a *is finite dimensional.*

Assume in addition that

($\mathscr{H}$). $D(A(t)) = D$-*independent of* t (*with* D *dense in* H), *and for each* $x \in D$ *the function* $t \to A(t)x$ *belongs to* $L^2(0,T;H)$. *Then*, *for* $a = 1$ *the following three additional properties hold* (*with* $M = M_1$).

5. M *is densely defined and* (*strongly*) *closed in* $L^2(0,T;H)$. (*Thus* $M^{**} = M$, *and* M *is weakly–weakly closed in* $L^2(0,T;H)$.)

6. $A^*(t) = -A(t)$ *for all* $t \in [0,T]$ *implies* $M^* = -M$ (*i.e. if each* $A(t)$ *is skew-adjoint*, *so is* M).

7. M *is not selfadjoint* (*even if all of* $A(t)$ *are selfadjoint*).

Proof. It is clear that (1.7) gives the following simple characterization of the graph and range of M_a, namely:

$$R(M_a) = \left\{ f \in L^2(0,T;X);\ \int_0^T U(T,s)f(s)\,\mathrm{d}s \in R(aI - U(T,0)) \right\} \tag{2.5}$$

and

$$M_a y = f \quad \text{iff } (aI - U(T,0))y(0) = \int_0^T U(T,s)f(s)\,\mathrm{d}s$$

and

$$y(t) = U(t,0)y(0) + \int_0^t U(t,s)f(s)\,\mathrm{d}s. \tag{2.6}$$

1. We prove only the implication

$$R(M_a) = L^2(0,T;X) \Rightarrow R(aI - U(T,0)) = X$$

(as its reverse is obvious, in view of (2.5) and (2.6)). Therefore, take $x \in X$ and prove the existence of an element $h \in X$ such that $(aI - U(T,0))h = x$. Indeed, for $f(s) = U(s,0)x$, the corresponding solution y_x satisfies (by (2.6) and $U(T,s)U(s,0) = U(T,0)$)

$$(aI - U(T,0))y_x(0) = \int_0^T U(T,s)f(s)\,\mathrm{d}s = TU(T,0)x$$

which can be written as $(aI - U(T,0))(a^{-1}T^{-1}y_x(0) + a^{-1}x) = x$.

2. Let $R(M_a) = L^2(0,T;X)$. If the solution y of $M_a y = f$ is unique, then (2.6) implies that $aI - U(T,0)$ is one to one. It is also onto so, $a \in \rho(U(T,0))$. In this case the unique solution of $M_a y = f$ is given by (in view of (2.6))

$$y(t) = U(t,0)(aI - U(T,0))^{-1} \int_0^T U(T,s)f(s)\,\mathrm{d}s + \int_0^t U(t,s)f(s)\,\mathrm{d}s. \tag{2.7}$$

3. We will prove only the implication

$R(M_a)$-closed in $L^2(0,T;X) \Rightarrow R(aI - U(T,0))$-closed in X (as its converse is obvious). Therefore, let $v_n = (aI - U(T,0))z_n$ with $v_n \to v$ in X. We prove that $v \in R(aI - U(T,0))$. To this goal, we first observe that the function f_n given by $f_n(s) = U(s,0)v_n$ belongs to $R(M_a)$ (on the basis of (2.5)). Indeed, $\int_0^T U(T,s)f_n(s)\,\mathrm{d}s = (aI - U(T,0))TU(T,0)z_n$. Taking into account that $f_n(s) \to f(s) = U(s,0)v$ and $R(M_a)$ is supposed to be closed, it follows that $\lim_{n\to\infty} f_n = f \in R(M_a)$, so $\int_0^T U(T,s)f(s)\,\mathrm{d}s = TU(T,0)v \in R(aI - U(T,0))$, i.e. there is $x \in X$ such that $TU(T,0)v = (aI - U(T,0))x$, which yields $v = (aI - U(T,0))(a^{-1}xT^{-1} + a^{-1}v)$.

4. If $U(T,0)$ is compact, then $R(aI - U(T,0))$ is closed in X so $R(M_a)$ is closed in $L^2(0,T;X)$. Moreover, the null space $N(aI - U(T,0))$ is finite dimensional, and therefore, in view of (2.6), $N(M_a)$ is also finite dimensional.

5. Clearly, $W^{1,2}_{T,A}(0,T;H) \subset D(M)$, as for such functions $M\psi = f$ means $(M\psi)(t) = \psi' - A(t)\psi = f(t)$. This is because a differentiable mild solution is a strong solution, too. On the other hand, $W^{1,2}_{T,A}(0,T;H)$ contains all functions of the form $\psi = \psi_m e^{\mu_m it}$ with $\psi_m \in D$. This implies that $W^{1,2}_{T,A}(0,T;H)$ (and consequently $D(M)$) is dense in

$L^2(0,T;H)$. Indeed, let $y \in L^2(0,T;H)$ be as in (1.11) and $\varepsilon > 0$. Choose an $y_m^\varepsilon \in D(A)$ such that $\|y_m - y_m^\varepsilon\| \leqslant \varepsilon 2^{-1} m^{-1} c^{-1}$ where $c = \sum_m m^{-2}$. Set

$$y_N^\varepsilon(t) = \sum_{m \in Z, |m| \leqslant N} y_m^\varepsilon e^{\mu_m i t} \tag{2.8}$$

and $y_N(t) = \sum_{|m| \leqslant N} y_m e^{\mu_m i t}$. Clearly, $y_N^\varepsilon \in W_{T,A}^{1,2}(0,T;H)$, and (in view of Parseval's formula), for sufficiently large N we have $\|y - y_N^\varepsilon\| \leqslant \varepsilon$. We now prove the key property, i.e. that M is closed in $L^2(0,T;H)$. Therefore, let $My_n = f_n$ with $y_n \to y$ and $f_n \to f$ in $L^2(0,T;H)$, i.e.

$$y_n(t) = U(t,0)y_n(0) + \int_0^t U(t,s) f_n(s)\, ds, \quad y_n(T) = y_n(0), \quad \forall t \in [0,T]. \tag{2.9}$$

It is not obvious that we can let $n \to \infty$ in (2.9), as we do not know that $y_n(0)$ is convergent in H. But we know that $y_n(t) \to y(t)$ in H for almost all $t \in [0,T]$. Let t_o be such a point, i.e. $y_n(t_o) \to y(t_o)$ and let us observe that (2.9) can be written in the form

$$y_n(t) = U(t,t_o)y_n(t_o) + \int_0^t U(t,s) f_n(s)\, ds, \quad t_o \leqslant t \leqslant T. \tag{2.10}$$

This implies that $y_n(t) \to y(t)$ for all $t_o \leqslant t \leqslant T$ so $\lim y_n(T) = \lim y_n(0) = y(T)$. Letting $n \to \infty$ in (2.9), one gets (2.6), i.e. $My = f$. Moreover, it follows that $y_n \to y$ in $C([0,T];H)$. We now prove that M is weakly–weakly closed (i.e. its graph $G(M)$ is weakly closed in $L^2(0,T;H)$). Therefore, let $My_n = f_n$ with $y_n \rightharpoonup y, f_n \rightharpoonup f$. Then $\langle y_n, M^* z\rangle = \langle f_n, z\rangle$, $\forall z \in D(M^*)$. Letting $n \to \infty$ and taking into account $M^{**} = M$, we get $My = f$. Set also

$$\begin{aligned} W_{T,A^*}^{1,2}(0,T;H) = \{\varphi \in W^{1,2}(0,T;H);\ &\varphi(0) = \varphi(T), \varphi(t) \in D = D(A^*(t)), \\ &\text{a.e. } t \in (0,T),\ t \to A(t)\varphi(t) \text{ is in } L^2(0,T;H)\}. \end{aligned} \tag{2.11}$$

Then $M\psi = f$ and $M^*\varphi = g$ yield (as a differentiable mild solution is a strong solution)

$$(M\psi)(t) = \psi' - A(t)\psi = f(t), \quad (M^*\varphi)(t) = -\varphi' - A^*(t)\varphi = g(t) \tag{2.12}$$

for $\psi \in W_{T,A}^{1,2}(0,T;H)$, $\varphi \in W_{T,A^*}^{1,2}(0,T;H)$ which implies the properties 6 and 7. This completes the proof. □

Remark 2.1. The density of $D(M_a)$ in $L^2(0,T;H)$ for $|a| \neq 1$, $a \neq 0$, as well as the density of M in $L^2(0,T;X)$, remain to be clarified.

It is now routine to check that the mild solution z to the problem

$$z' = -A^*(t)z - g, \quad z(T) = z(0) \tag{2.13}$$

(i.e. $M^*z = g$), is given by

$$z(t) = U^*(T,t)z(T) + \int_t^T U^*(s,t)g(s)\, ds, \quad z(T) = z(0), \quad \forall t \in [0,T]. \tag{2.14}$$

Remark 2.2. If the range $R(M)$ (of the closed densely defined linear operator M) is closed in $L^2(0,T;H)$, then we have

$$L^2(0,T;H) = N(M) \oplus R(M^*) = R(M) \oplus N(M^*) \tag{2.15}$$

$$H = N(I - U(T,0)) \oplus R(I - U^*(T,0)) = N(I - U^*(T,0)) \oplus R(I - U(T,0)) \tag{2.16}$$

and the restriction $\tilde{M}$ of M, $\tilde{M} : D(M) \cap R(M^*) \to R(M)$ is one-to-one and onto, so it is invertible. Its inverse $\tilde{M}^{-1} : R(M) \to D(M) \cap R(M^*) \subset R(M^*)$ is closed (i.e. its graph is closed in $L^2(0,T;H)$) so, in view of the closed graph theorem, $\tilde{M}^{-1}$ is continuous on $R(M)$ (i.e. $\tilde{M}^{-1} \in L(R(M);R(M^*))$). Clearly, $\tilde{M}$ is densely defined in $R(M^*)$. Similarly $(I - U(T,0))^{-1} : R(I - U(T,0)) \to R(I - U^*(T,0))$ is bounded in H.

Proposition 2.1. *Suppose that $U(t,s)$ is compact in X for every pair (t,s) with $t>s$, $1 \in \rho(U(T,0))$ and $U(t,s)$ is continuous in the uniform operator topology on $0 \leqslant s < t \leqslant T$. Then M^{-1} is a compact operator from $L^2(0,T;X)$ into $C([0,T];X)$ (i.e. it maps bounded subsets of $L^2(0,T;X)$ into relatively compact subsets of $C([0,T];X)$.*

Proof. Let E be a bounded subset of $L^2(0,T;X)$ and let $f \in E$. Then $My_f = f$ means that

$$y_f(t) = U(t,0)(I - U(T,0))^{-1} \int_0^T U(T,s)f(s)\,\mathrm{d}s + \int_0^t U(t,s)f(s)\,\mathrm{d}s. \tag{2.17}$$

We will prove that $M^{-1}E = \{y_f; f \in E\}$ is relatively compact (precompact) in the strong topology of $C([0,T];X)$. To this aim one checks that the functions in $M^{-1}E$ are equicontinuous (and uniformly bounded in $C([0,T];X)$), by using the fact that $t \to U(t,s)$, $t > s$ are continuous in the uniform operator topology. In addition we prove that for each $t \in [0,T]$, the set $S_t = \{y_f(t); f \in E\}$ is relatively compact in $C([0,T];X)$. For $t = 0$ it suffices to show that the subset $K = \{x_f = \int_0^T U(T,s)f(s)\,\mathrm{d}s, f \in E\}$ is relatively compact in X. To this aim, let $\varepsilon > 0$ and $K_\varepsilon = \{\int_0^{T-\varepsilon} U(T,s)f(s)\,\mathrm{d}s; f \in E\}$. This is relatively compact in X due to the equality

$$\int_0^{T-\varepsilon} U(T,s)f(s)\,\mathrm{d}s = U(T,T-\varepsilon)\int_0^{T-\varepsilon} U(T-\varepsilon,s)f(s)\,\mathrm{d}s$$

and to the compactness of $U(T,T-\varepsilon)$. Now let

$$x_f \in K \quad \text{and} \quad x_\varepsilon = \int_0^{T-\varepsilon} U(T,s)f(s)\,\mathrm{d}s \in K_\varepsilon.$$

Clearly, $x_f - x_\varepsilon = \int_{T-\varepsilon}^T U(T,s)f(s)\,\mathrm{d}s$ so there is a constant c such that $\|x - x_\varepsilon\| \leqslant c\varepsilon$. This implies the relative compactness of K. Therefore S_0 is relatively compact in X. With similar arguments one proves the precompactness of each S_t. By Ascoli–Arzela criterion, $M^{-1}E$ is compact in $C([0,T];X)$, which completes the proof. $\square$

2.1. Basic properties of $\tilde{M}$ (see Remark 2.2)

Let $My = f$. Then y can be uniquely written as

$$y = y_f^N + y_f^R \quad \text{with } y_f^N \in N(M),\ y_f^R \in R(M^*), \text{ i.e. } \tilde{M} y_f^R = f \tag{2.18}$$

with

$$N(M) = \{y^N, y^N(t) = U(t,0)y^N(0); (I - U(T,0))y^N(0) = 0, \\ \text{i.e. } y^N(0) \in N(I - U(T,0))\} \tag{2.19}$$

and

$$(\tilde{M}^{-1} f)(t) = y_f^R(t) = U(t,0)y_f^R(0) + \int_0^t U(t,s)f(s)\,\mathrm{d}s, \quad f \in R(M)$$

$$(I - U(T,0))y_f^R(0) = \int_0^T U(T,s)f(s)\,\mathrm{d}s. \tag{2.20}$$

In other words, $y^N \in N(M)$ implies $y^N(0) \in N(I - U(T,0))$ and vice versa. One would expect $y_f^R \in R(M^*)$ to imply $y_f^R(0) \in R(I - U^*(T,0))$. But this is not true, as shown in the example below. Indeed, let us consider the simplest possible case in $L^2(0,T;R)$, namely

$$My = y' - Ay \quad \text{with } A = 0 = A^*, \quad \text{so } My = y',\ M^*z = -z' \tag{2.21}$$

with the T-periodic condition $y(0) = y(T)$, $z(0) = z(T)$.

Obviously the T-periodic solution of equation $y'(t) = f(t), y(0) = y(T)$ is given by

$$y(t) = y(T) + \int_0^t f(s)\,\mathrm{d}s \quad \text{so} \int_0^T f(s)\,\mathrm{d}s = 0. \tag{2.22}$$

Therefore

$$R(M) = R(M^*) = \left\{ f \in L^2(0,T;R); \int_0^T f(s)\,\mathrm{d}s = 0 \right\}, \quad N(M) = \tilde{R}$$

$$A = 0, \quad 1 - \mathrm{e}^{AT} = 0, \quad R(1 - \mathrm{e}^{0T}) = \{0\}, \quad N(1 - \mathrm{e}^{0T}) = R, \tag{2.23}$$

where $\tilde{R}$ is the set of all constant functions on R. Clearly, the above solution y can be uniquely written as $y(t) = a + y_f^R(t)$ with $\int_0^T y_f^R(s)\,\mathrm{d}s = 0$ and $a \in R$ (i.e. $a \in N(M) = \tilde{R}$ and $y_f^R \in R(M^*)$). Thus $y_f^R(t) = y(0) - a + \int_0^t f(s)\,\mathrm{d}s$ which yields (integrating over $[0,T]$) $y(0) - a = -T^{-1}\int_0^T \int_0^t f(s)\,\mathrm{d}s\,\mathrm{d}t$. Therefore the unique component of y in $R(M^*)$ is given by

$$(\tilde{M}^{-1} f)(t) = y_f^R(t) = \int_0^t f(s)\,\mathrm{d}s - T^{-1}\int_0^T \int_0^t f(s)\,\mathrm{d}s\,\mathrm{d}t, \quad f \in R(M) \tag{2.24}$$

so

$$y_f^R(0) = -T^{-1}\int_0^T \int_0^t f(s)\,\mathrm{d}s\,\mathrm{d}t = T^{-1}\int_0^T s f(s)\,\mathrm{d}s. \tag{2.25}$$

We do have in this case

$$\|\tilde{M}^{-1} f\|_{C([0,T];R)} \leqslant c\|f\|_{L^1(0,T;R)}, \quad \forall f \in R(M). \tag{2.26}$$

However, $y_f^R(0)$ may be different from zero. Indeed, for $T=\pi$, the function $f(s)=\cos s$ belongs to $R(M)$, with $y_f^R(0)=-2\pi^{-1}$ which does not belong to $R(1-e^{0T})=\{0\}$. In other words $\tilde{M}y_f^R=f$ (which means $My_f^R=f$ and $y_f^R\in R(M^*)$), does not imply $y_f^R(0)\in R(I-U^*(T,0))$ even in the autonomous case $A(t)=A$-independent of t. Let us see the consequences of this fact. As $y_f^R(0)$ may not belong to $R(I-U^*(T,0))$, the representation (see (2.20))

$$(\tilde{M}^{-1}f)(0)=y_f^R(0)=(I-U(T,0))^{-1}\int_0^T U(T,s)f(s)\,\mathrm{d}s \tag{2.27}$$

is not valid, as $(I-U(T,0))^{-1}:R(I-U(T,0))\to R(I-U^*(T,0))$ (Remark 2.2). However, the inequality below holds (a general case of (2.26))

$$\|\tilde{M}^{-1}f\|_{C([0,T];H)}\leqslant c\|f\|_{L^1(0,T;H)},\quad \forall f\in R(M). \tag{2.28}$$

This inequality is very important in control problems (Barbu [3]) and in itself. In the autonomous case it is due to Barbu [3]. Moreover, the proof of this inequality is due to Barbu and Pavel. It is valid in the nonautonomous case, too, as shown below. The proof is based on the continuity of the projection operator P on $R(M^*)$ and on the lemma below.

Lemma 2.1. *Let $My_n=f_n$ be such that $y_n\to y$ in $L^2(0,T;H)$ and $f_n\to f$ in $L^1(0,T;H)$. Then $y_n(0)\to y(0)$ in H, $My=f$ (i.e. M is closed in $L^2(0,T;H)\times L^1(0,T;H)$) and $y_n\to y$ in $C([0,T];H)$.*

Proof. One uses (2.9) and (2.10) with $f_n\to f$ in $L^1(0,T;H)$ in place of $f_n\to f$ in $L^2(0,T;H)$.

Proof of Inequality (2.28). For $f\in R(M)$, $(\tilde{M}^{-1}f)(t)=y_f^R(t)$ is given by (2.20).

On the other hand, $y_f^R(T)=y_f^R(0)$ can be uniquely written as

$$y_f^R(0)=a_f^N+b_f^R\quad\text{with } a_f^N\in N(I-U(T,0)),\ b_f^R\in R(I-U^*(T,0)). \tag{2.29}$$

Substituting into (2.20) and rearranging we get

$$\int_0^t U(t,s)f(s)\,\mathrm{d}s+U(t,0)b_f^R=y_f^R(t)-U(t,0)a_f^N \tag{2.30}$$

with $y_f^R\in R(M^*)$ and $t\to U(t,0)a_f^N$ in $N(M)$. Set

$$(Df)(t)=\int_0^t U(t,s)f(s)\,\mathrm{d}s,\quad (Ef)(t)=U(t,0)b_f^R. \tag{2.31}$$

Therefore (2.29), (2.30), (2.20) and $f\in R(M)$, lead to

$$P(Df+Ef)=(\tilde{M})^{-1}f,\quad b_f^R=(I-U(T,0))^{-1}\int_0^T U(T,s)f(s)\,\mathrm{d}s, \tag{2.32}$$

where P is the projection operator on $R(M^*)$. Clearly, Df and Ef are continuous from $L^1(0,T;H)$ into $L^2(0,T;H)$, therefore, on the basis of the continuity of P on $L^2(0,T;H)$, (2.32) implies the continuity of $(\tilde{M})^{-1}$ from $L^1(0,T;H)$ into $L^2(0,T;H)$.

Finally, this and Lemma 2.1 lead to the continuity of $f \to y_f(0) = (\tilde{M}^{-1} f)(0)$ from $L^1(0,T;H)$ into H, i.e. there is a constant d such that

$$\|(\tilde{M}^{-1} f)(0)\|_H \leqslant d\|f\|_{L^1(0,T;H)}, \quad \forall f \in R(M). \tag{2.33}$$

This in conjunction with (2.20) imply inequality (2.28), and the proof is complete. □

It is interesting that Proposition 2.1 (from the nonresonant case in Banach spaces) on the compactness of M^{-1}, admits a natural extension to the nonresonant case (to Hilbert spaces, only), i.e. the compactness of $\tilde{M}^{-1}$. Precisely, we have:

Proposition 2.2. *Suppose that $U(t,s)$ is compact in H for every pair (t,s) with $t > s$, and $U(t,s)$ is continuous in the uniform operator topology on $0 \leqslant s < t \leqslant T$. Then $\tilde{M}^{-1}$ is a compact operator from $L^2(0,T;H)$ into itself (i.e. it maps bounded subsets of $L^2(0,T;H)$ into relatively compact subsets of $L^2(0,T;H)$).*

Proof. Arguing as in the proof of Proposition 2.1, it is clear that the operators D and E in (2.31) and (2.32) are compact from $L^2(0,T;H)$ into $C([0,T];H)$. This, the key formula (2.32) and the continuity of P in $L^2(0,T;H)$, completes the proof. □

Remark 2.3. The simplest t-dependent case but still important, entirely covered by Proposition 2.2, is the finite dimensional case $H = R^m$ with $A(t)$-$m \times m$ matrices with continuous elements.

3. Necessary conditions of optimality

We start with the following remarkable simple and useful optimality principle in terms of Clarke's generalized gradients.

Lemma 3.1. *Let D be a nonempty subset of a Banach space X, D_o an open neighborhood of D and let $F : D_o \to R$ be a locally Lipschitz real-valued function on D_o. Suppose that*

$$\operatorname*{Inf}_{x \in D} F(x) = F(x_o), \quad x_o \in D$$

and that the contingent cone $K_D(x_o)$ to D at x_o is a subspace of X. Then there exists an element $\xi \in \partial F(x_o)$ (the generalized gradient of F at x_o in the sense of Clarke) such that $\langle \xi, v\rangle = 0$, for all $v \in K_D(x_o)$.

We will sketch its (surprisingly simple) proof in order to make it independent of more general results of this type.

Proof of Lemma 3.1. First of all let us recall the definitions of: the generalized directional derivative $F^o(x;v)$ of F at x in the direction of v, the generalized gradient $\partial F(x)$

of F at x and of the contingent (tangential) cone $K_D(x)$

$$F^{\mathrm{o}}(x;v) = \limsup_{\substack{z\to x\\ t\downarrow 0}} \frac{F(z+tv)-F(z)}{t},$$

$$\partial F(x) = \{\xi \in X^*; \langle \xi, v\rangle \leqslant F^{\mathrm{o}}(x;v), \forall v \in X\},$$

$$K_D(x) = \{v \in X; \exists r(t) \to 0 \text{ as } t \downarrow 0, \text{such that } x_t = x + tv + tr(t) \in D\}. \tag{3.1}$$

If $v \in K_D(x_{\mathrm{o}})$, then there is $r(t) \to 0$ when $t \downarrow 0$ such that $(x_{\mathrm{o}} + tv + tr(t)) \in D$, so

$$F(x_{\mathrm{o}} + tr(t) + tv) - F(x_{\mathrm{o}}) \geqslant 0, \quad \text{for all sufficiently small } t \geqslant 0. \tag{3.2}$$

Set $z_t = x_{\mathrm{o}} + tr(t)$. The Lipschitz continuity of F near x_{o} implies

$$\lim_{t\downarrow 0} \frac{F(z_t) - F(x_{\mathrm{o}})}{t} = 0$$

and therefore (3.2) yields

$$F^{\mathrm{o}}(x_{\mathrm{o}}; v) \geqslant 0, \quad \forall v \in K_D(x_{\mathrm{o}}). \tag{3.3}$$

Denote by g the null functional on $K_D(x_{\mathrm{o}})$, so by (3.3)

$$g(v) \leqslant F^{\mathrm{o}}(x_{\mathrm{o}}; v), \quad \forall v \in K_D(x_{\mathrm{o}}). \tag{3.4}$$

As $v \to F^{\mathrm{o}}(x_{\mathrm{o}}; v)$ is subadditive and positively homogeneous on X, the Hahn–Banach theorem guarantees that g can be extended to the whole X by preserving (3.4). Denoting by ξ such an extension of g, one concludes the proof. □

In applications to control theory the following type of "cost functionals" are usually involved:

$$L(y,u) = G(y) + J(u), \quad y \in X_1, u \in X_2, X_n\text{-Banach spaces}, \ n = 1,2. \tag{3.5}$$

In this case Lemma 3.1 yields

Corollary 3.1. *Let G and J above be local Lipschitz functionals and let D be a nonempty subset of the Cartesian product $X_1 \times X_2$. Suppose that*

$$\inf\{L(y,u); (y,u) \in D\} = L(\bar{y}, \bar{u})$$

and that $K_D(\bar{y},\bar{u})$ is a subspace of $X_1 \times X_2$. Then there are $(\xi,\eta) \in \partial L(\bar{y},\bar{u})$, with $\xi \in \partial G(\bar{y})$ and $\eta \in \partial J(\bar{u})$ such that

$$\langle \xi, v\rangle + \langle \eta, w\rangle = 0, \quad \forall (v,w) \in K_D(\bar{y},\bar{u}). \tag{3.6}$$

Proof. On the basis of Lemma 3.1, there is $(\xi,\eta) \in \partial L(\bar{y},\bar{u})$ satisfying (3.6). On the other hand we have

$$\langle \xi, v\rangle + \langle \eta, w\rangle \leqslant L^{\mathrm{o}}((\bar{y},\bar{u});(v,w)) \leqslant G^{\mathrm{o}}(\bar{y}; v) + J^{\mathrm{o}}(\bar{u}; w)$$

for all $v \in X_1$ and $w \in X_2$. For $w = 0$ it implies $\xi \in \partial G(\bar{y})$, and similarly $\eta \in \partial J(\bar{u})$, which completes the proof. □

In applications to optimal control of PDE the following result (proven by Clarke [9, pp. 75–80] in more general conditions) plays an essential role.

Proposition 3.1. *Let $g : R \to R$ be a Lipschitz function and Ω a bounded domain of R^n. Consider the functional*

$$G(y) = \int_\Omega g(y(x))\,\mathrm{d}x, \quad y \in L^2(\Omega). \tag{3.7}$$

Then for every $\xi \in \partial G(y)$, there is an L^2-selection ζ of $\partial g(y(x))$ (i.e. $\zeta(x) \in \partial g(y(x))$ a.e. on Ω) such that

$$\langle \xi, v\rangle = \int_\Omega \langle \zeta(x), v(x)\rangle\,\mathrm{d}x, \quad \forall v \in L^2(\Omega). \tag{3.8}$$

One can identify ξ with ζ.

We are now in a position to give necessary conditions for an admissible pair $(\bar{y}, \bar{u})$ to be optimal. Precisely, consider the problem

(P_1) Minimize: $L(y,u) = G(y) + J(u)$

over all (y,u) constrained to

$$y(t) = U(t,0)y(0) + \int_0^t U(t,s)((Bu)(s) + f(s))\,\mathrm{d}s, \quad \forall t \in [0,T], \quad y(0) = y(T). \tag{3.9}$$

In terms of the weak solution operator M the constrain (3.9) means that (y,u) belongs to

$$D = \{(y,u); y \in C([0,T];H), u \in L^2(0,T;H), My = Bu + f\}. \tag{3.10}$$

The hypotheses are the following:

(H1) The functionals G and J are local Lipschitz. $B : U \subset X \to H$ is a Fréchet differentiable function on an open subset U of a Banach space X and in addition: For each $u \in U$, the range $R(B'(u))$ is closed in $L^2(0,T;H)$.

(H2) $U(t,s)$ is an evolution operator generated by a family of linear operators $\{A(t)\}$ acting in H, with $D(A(t))=D$-independent of t, D dense in H and $R(I-U(T,0))$-closed in H. For each $x \in D$ the function $t \to A(t)x$ belongs to $L^2(0,T;H)$. $\|U(t,s)\| \leqslant c$, $0 \leqslant s \leqslant t \leqslant T$.

(H3) For each $w \in L^2(0,T;H)$, there is $r(t) \to 0$ in X as $t \to 0$ such that

$$B(u) + tB'(u)w = B(u + tw + tr(t)), \quad \forall t \text{ sufficiently small, } u \in U. \tag{3.11}$$

(H3)′ $R(B) \subset R(M)$.

The main result of this section is the following one.

Theorem 3.1. (I) *Assume that Hypotheses* H(1)–H(2) *are fulfilled and the pair $(\bar{y}, \bar{u}) \in D$ is optimal, i.e.*

$$Inf\{L(y,u); (y,u) \in D\} = L(\bar{y}, \bar{u}).$$

Assume in addition that one of the inclusions $R(M) \subset R(B'(\bar{u}))$ or $R(B'(\bar{u})) \subset R(M)$ holds and that either (H3) *or* (H3)′ *is fulfilled. Then there is a $p \in D(M^*)$ such that*

$$M^* p \in -\partial G(\bar{y}), \quad (B'(\bar{u}))^* p \in \partial J(\bar{u}). \tag{3.12}$$

(II) *If B is a closed densely defined linear operator with closed range $R(B)$ and one of the inclusions $R(M) \subset R(B)$ or $R(B) \subset R(M)$ holds, then there is a $p \in D(M^*) \cap D(B^*)$ such that*

$$M^* p \in -\partial G(\bar{y}), \quad B^* p \in \partial J(\bar{u}). \tag{3.13}$$

In our case here, the first inclusion in (3.13) means the existence of an element $\xi \in \partial G(\bar{y})$ such that (see (2.14))

$M^* p = -\xi$, i.e.

$$p(t) = U^*(T,t)p(T) - \int_t^T U^*(s,t)\xi(s)\,\mathrm{d}s, \quad p(T) = p(0), \quad \forall t \in [0,T]. \tag{3.14}$$

In other words, p is a mild solution to the problem

$$z' = -A^*(t)z + \xi, \quad z(T) = z(0). \tag{3.14$'$}$$

The formula (3.14) is known in the convex case (i.e. L convex, see Barbu and Precupanu [7]). In this case the generalized gradient coincide with the subdifferential of L (cf. Clarke [9]).

In order to have precise informations on the contingent cone $K_D(y,u)$ to the constraints set D (at $(y,u) \in D$ given in (3.10)), we first deal with

Lemma 3.2. (1) *Let M and B be linear operators. Then*

$$\{(v,w); Mv = Bw\} \subset K_D(y,u), \quad \forall (y,u) \in D.$$

(2) *Let $B: U \subset X \to H$ be a Fréchet differentiable function on an open subset U of a Banach space X and let M be a closed linear operator. Then*

$$K_D(y,u) \subset \{(v,w); Mv = (B'(u))w\} \doteq K_1. \tag{3.15}$$

(3) *Suppose in addition to* (2), *that $R(M)$ is closed and either* (H3) *or* (H3)$'$ *holds. Then*

$$K_D(y,u) = \{(v,w); Mv = (B'(u))w\} \doteq K_1. \tag{3.15$'$}$$

Proof. (1) is immediate as $My = Bu + f$ and $Mv = Bw$ yield $M(y+tv) = B(u+tw) + f$ for all t (i.e. $(y+tv, u+tw) \in D$, so $(v,w) \in K_D(y,u)$).

(2) Let $(v,w) \in K_D(u,v)$. This means the existence of $r_j(t) \to 0$, $j = 1,2,3$ such that

$$M(y + tv + tr_1(t)) = B(u + tw + tr_2(t)) + f = B(u) + t(B'(u))(w) + tr_3(t) + f$$

so $M(v + r_1(t)) = (B'(u))(w) + r_3(t)$. Letting $t \downarrow 0$ and taking into account that M is closed we get $Mv = (B'(u))(w)$, so (3.15) holds.

(3) If (H3) holds, then $K_1 \subset K_D(y,u)$ follows easily. Indeed, (3.11) implies that if $(v,w) \in K_1$, then

$$(y + tv, u + tw + tr(t)) \in D, \quad \text{so } (v,w) \in K_D(y,u).$$

If (H3)$'$ holds, then the remainder $R(t)$ in the definition of $(B'(u))(w)$, precisely, $B(u+tw) = B(u) + t(B'(u))(w) + tR(t)$ with $(v,w) \in K_1$ satisfies $R(t) \in R(M)$. Therefore there is $a(t)$ such that $Ma(t) = R(t)$. Since $R(M)$ is closed we take the projection $r(t)$ of $a(t)$ on $R(M^*)$, i.e. $r(t) = \tilde{M}^{-1}R(t) \to 0$ (as $\tilde{M}^{-1}$ is continuous-see Remark 2.5). Clearly, this means $M(r(t)) = R(t)$ which implies that $(y + tv + tr(t), u + tw) \in D$ so $(v, w) \in K_D(y, u)$, which completes the proof. □

Proof of Theorem 3.1. Hypotheses (H2) guarantee that the linear operator M is closed, densely defined with closed range $R(M)$ in $L^2(0, T; H)$ as seen by the end of the previous section. Moreover, (3.11) assures that the contingent cone $K_D(\bar{y}, \bar{u})$ to D at $(\bar{y}, \bar{u})$ is given by (3.15)′ with $(\bar{y}, \bar{u})$ in place of (y, u). From now on the proof proceeds as in the proof of Theorem 1 from [1]. However, in our framework here, the proof is simplified so we will sketch it. On the basis of Corollary 3.1 and Lemma 3.2, there are $(\xi, \eta) \in \partial L(\bar{y}, \bar{u})$, with $\xi \in \partial G(\bar{y})$ and $\eta \in \partial J(\bar{u})$ such that

$$\langle \xi, v\rangle + \langle \eta, w\rangle = 0, \quad \forall (v, w) \in K_D(\bar{y}, \bar{u}). \tag{3.16}$$

For $w = 0$ it implies that $\xi \in (N(M))^{\perp} = R(M^*)$ (as $R(M)$ is closed in $L^2(0, T; H)$, see e.g. Brezis [8]), so there is $p_{\rm o}$ such that $\xi = M^* p_{\rm o}$. Similarly (for $v = 0$) we conclude that there is a p such that $\eta = (B'(\bar{u})^*)p$. Say that $R(M) \subset R(B'(\bar{u}))$. Going back to (3.16) we get $\langle p_{\rm o} + p, Mv\rangle = 0$ for all $v \in D(M)$ so $(p_{\rm o} + p) \in (R(M))^{\perp} = N(M^*)$ hence $M^* p_{\rm o} = -M^* p$, so (3.12) holds. Similarly one proves (3.13), which concludes the proof. □

3.1. Existence of optimal pairs

The existence of optimal pairs corresponding to nonconvex cost functionals (and even worse nonlinear constraints) is a difficult and little known problem. In general in such cases, optimal pairs may not even exist. However, we do have important cases of nonconvex cost functionals associated with resonant systems, in which we can prove (on the basis of the key result given in Proposition 2.2) the existence of optimal pairs:

Theorem 3.2. *Let $G : L^2(0, T; H) \to R$ be a continuous functional and let $J : L^2(0, T; H) \to \bar{R}$ be a lower semicontinuous convex functional satisfying the growth conditions (for some constants a,b):*

$$a\|y\| + b \leqslant G(y), \quad a_1\|y\| + b_1 \leqslant J(u), \quad y, u \in L^2(0, T; H), \;\; a, a_1 > 0. \tag{3.17}$$

Assume in addition that $U(t, s)$ is (compact) as indicated in Proposition 2.2. Then the problem (P_1) with B, a linear bounded operator on $L^2(0, T; H)$, admits optimal pairs $(\bar{y}, \bar{u})$.

Proof. Set $d = \mathrm{Inf}\{G(y) + J(u); (y, u) \in D\}$ with D as in (3.11) and let (y_n, u_n) be a minimizing sequence, i.e.,

$$My_n = Bu_n + f, \quad \lim_{n\to\infty} [G(y_n) + J(u_n)] = d, \quad G(y_n) + J(u_n) \leqslant d + n^{-1} \tag{3.18}$$

(for n sufficiently large) which implies (in conjunction with (3.17)) the boundedness of y_n and u_n in $L^2(0,T;H)$. We may assume, relabeling if necessary, that y_n and u_n are weakly convergent to $\bar{y}$ and $\bar{u}$, respectively. But y_n can be uniquely written as

$$y_n = y_n^N + y_n^R \quad \text{with } y_n^N \in N(M), \quad y_n^R \in R(M^*), \text{ i.e. } \tilde{M} y_n^R = Bu_n + f. \tag{3.19}$$

On the basis of Proposition 2.2, $\tilde{M}^{-1}$ is compact, so $y_n^R = \tilde{M}^{-1}(Bu_n + f)$ is relatively compact in $L^2(0,T;H)$. This implies the boundedness of y_n^N in the finite dimensional space $N(M)$. It follows that y_n contains a strongly convergent sequence, denoted again by y_n. Letting $n \to \infty$ in (3.18) one derives $G(\bar{y},\bar{u}) + J(\bar{y},\bar{u}) = d$. This and the fact that D is weakly closed, complete the proof. □

Remark 3.1. In principle, the existence of optimal control of t-dependent parabolic equations (with $D(A(t))$ independent of t) with time-periodic conditions, in Sobolev spaces, fits into this framework. This is mainly because they can be treated via compact semigroups and evolution operators [7,16]. However, the above maximum principles (Theorem 3.1) can be applied to hyperbolic and elliptic equations, too. A sample of applications to the parabolic case (including the existence of optimal pairs) is the following one:

Let $g : R \to R$ and $j : R \to R$ be functions such that their L^2 integrand G and J are locally Lipschitz, satisfying the conditions above. Recall that G is precisely defined by $G(y) = \int_Q g(y(t,x))\,\mathrm{d}t\,\mathrm{d}x$ and similarly $J(y) = \int_Q j(y(t,x))\,\mathrm{d}t\,\mathrm{d}x$. Let $\phi : [0,T] \to (0,+\infty)$ be a continuous and positive-valued function. Then the problem

$$\text{(Locally) Minimize} \quad \int_Q (g(y(t,x)) + j(u(t,x)))\,\mathrm{d}t\,\mathrm{d}x$$

subject to

$$y_t - \phi(t)\Delta_x y = u(t,x) + f(t,x), \quad t \in (0,T), \ x \in \Omega,$$

$$y(t,x) = 0, \quad \text{on } (0,T) \times \partial\Omega, \quad y(0,x) = y(T,x) \tag{3.20}$$

has a solution (optimal pair) (y^*, u^*). Moreover, a pair (y^*, u^*) is optimal iff there is p^* such that

$$p_t^* - \phi(t)\Delta p^* = -w(t,x),$$

$$p^*(t,x) = 0 \quad \text{on } (0,T) \times \partial\Omega, \quad p^*(0,x) = p^*(T,x),$$

$$w(t,x) \in \partial g(y^*(t,x)), \quad p^*(t,x) \in \partial j(u^*(t,x)) \quad \text{a.e. in } Q \tag{3.21}$$

or equivalently

$$u^*(t,x) \in \partial j^*(p^*(t,x)) \quad \text{a.e. in } Q, \tag{3.22}$$

where j^* is the conjugate of j, i.e.,

$$j^*(r) = \sup\{rv - j(v);\ v \in R,\ r \in R\}.$$

Indeed, the operators $A(t)y = \phi(t)\Delta y$ with $D(\Delta) = H_o^1(\Omega) \cup H^2(\Omega)$, generates a compact evolution operator $U(t,s)$, $t > s$. Indeed, by using the Fourier analysis in $L^2(\Omega)$ one proves that the set

$$K_t = \{U(t,s)z;\ \text{z in a bounded set B of } L^2(\Omega)\}$$

is bounded in $H_o^1(\Omega)$, so it is relatively compact in $H = L^2(\Omega)$. Moreover, $t \to U(t,s)z$ is equicontinuous on B at $t_o > s$. Precisely, one can prove the inequality:

$$\|U(t,0)z - U(t_o,0)z\| \leqslant \frac{d}{cet_o}|t - t_o|\,\|z\|, \quad z \in B \tag{3.23}$$

with $t_o > 0,\ 0 < c \leqslant \phi(t) \leqslant d,\ t \in [0,T]$.

Proof of Inequality (3.23). Set $y(t,z) = U(t,0)z$ and let $y_k(t)$ and z_k be the Fourier coefficients of y and z, respectively, with respect to the orthonormal basis $\{\psi_k\}$ of the eigenfunctions of the above Laplace operator in $L^2(\Omega)$, i.e., $\Delta\psi_k = -\lambda_k^2\psi_k, \lambda_k < \lambda_{k+1}$, $k = 1, \dots$. Then we have:

$$y_k'(t) = -\lambda_k^2\phi(t)y_k(t), y_k(0) = z_k, \text{ so, } y_k(t) = z_k e^{-\lambda_k^2 \int_0^t \phi(s)\,ds}.$$

Accordingly

$$y_k(t) - y_k(t_o) = (t - t_o)y_k'(\tau) = -\lambda_k^2(t - t_o)\phi(\tau)y_k(\tau),\ 0 < t_o < \tau < t$$

so

$$y_k(t) - y_k(t_o) = -\lambda_k^2(t - t_o)\phi(\tau)z_k e^{-\lambda_k^2 \int_0^\tau \phi(s)\,ds}.$$

This in conjunction with the elementary inequality $te^{-t} \leqslant e^{-1}$, $\forall t > 0$, yield (3.23). Thus, Theorems 3.1 and 3.2 lead to (3.21).

Remark 3.2. The case $D(A(t))$ t-dependent remains open in this framework. Recent results via Fourier analysis (which may be inspiring for an extension of this paper), are given in [15].

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