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Existence and Regularity for a Class of Nonlinear Hyperbolic Boundary Value Problems[1]

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The regularity of the solution of the telegraph system with nonlinear monotone boundary conditions is investigated by two methods. The first one is based on D'Alembert-type representation formulae for the solution. In the second method the telegraph system is reduced to a linear Cauchy problem with a locally Lipschitzian functional perturbation; then regularity results are established by appealing to the theory of linear semigroups. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Let $T > 0$, $D_T =]0, T[\times]0, 1[$ and denote $\partial u/\partial x = u_x$, $\partial u/\partial t = u_t$, etc. We study the following boundary value problem (BVP):

$$\begin{aligned} u_t(t,x) + v_x(t,x) + Ru(t,x) &= f_1(t,x), &\quad \text{for } (t,x) \in D_T, \\ v_t(t,x) + u_x(t,x) + Gv(t,x) &= f_2(t,x), &\quad \text{for } (t,x) \in D_T, \end{aligned} \tag{S}$$

$$(-u(t,0), u(t,1)) \in \beta(v(t,0), v(t,1)), \qquad \text{for } 0 < t < T, \tag{BC}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \qquad \text{for } 0 < x < 1, \tag{IC}$$

where $R, G \in \mathbf{R}, \beta \subset \mathbf{R}^2 \times \mathbf{R}^2$ is a maximal monotone operator; u_0, v_0: $[0,1] \mapsto \mathbf{R}$; and f_1, f_2: $[0,T] \times [0,1] \mapsto \mathbf{R}$. Clearly, (S) is the well-known telegraph system, and (BC) includes as particular cases various classical boundary conditions [5, Chap. III]. For an existence theory for our BVP, see [5, Chap. III], where the theory of evolution equations associated with monotone operators is used. We introduce two notions and recall two existence results. Let $f_1, f_2 \in L^1(0,T;L^2(0,1))$. A couple (u,v) of functions from $W^{1,1}(0,T;L^2(0,1))$ is called a *strong solution* of (S)–(BC), if $u_x, v_x \in L^1(0,T;L^2(0,1))$ and (u,v) satisfies (S) and (BC) almost everywhere. A *weak solution* of (S)–(BC) is a couple (u,v) of continuous $L^2(0,1)$-valued functions on $[0,T]$ if there exist functions $f_{1n}, f_{2n} \in L^1(0,T;L^2(0,1))$ and strong solutions (u_n, v_n) of (S)–(BC) with (f_{1n}, f_{2n}) instead of (f_1, f_2), for $n = 1, 2, \ldots,$ and

$$\lim_{n\to\infty}\left(\|u_n - u\|_{C([0,T];L^2(0,1))} + \|v_n - v\|_{C([0,T];L^2(0,1))}\right) = 0,$$
$$\lim_{n\to\infty}\left(\|f_{1n} - f_1\|_{L^1(0,T;L^2(0,1))} + \|f_{2n} - f_2\|_{L^1(0,T;L^2(0,1))}\right) = 0.$$

If (u,v) is a strong or weak solution of (S)–(BC) and it satisfies (IC) a.e. on $]0,1[$, then (u,v) is said to be a *strong* or *weak solution of our BVP*, respectively.

THEOREM 1.1. *Let $f_1, f_2 \in W^{1,1}(0,T;L^2(0,1))$ and $u_0, v_0 \in H^1(0,1)$ satisfy*

$$(-u_0(0), u_0(1)) \in \beta(v_0(0), v_0(1)). \tag{1.1}$$

Then our BVP has a unique strong solution $(u,v) \in W^{1,\infty}(0,T;L^2(0,1))^2$ with $u_x, v_x \in L^\infty(0,T;L^2(0,1))$.

THEOREM 1.2. *If $f_1, f_2 \in L^1(0,T;L^2(0,1))$ and $u_0, v_0 \in L^2(0,1)$, then our BVP has a unique weak solution $(u,v) \in C([0,T];L^2(0,1))^2$.*

We shall use two different approaches to derive existence and regularity results for our BVP: D'Alembert's representation formulae and the linear semigroup theory (see, e.g., [4, 6]). In particular, we improve a C^1-regularity result established in [1].

In Section 2 we write D'Alembert's formulae for our BVP with $R = G = 0$ and define the generalized solution. In Section 3 we briefly consider the case where R or G does not vanish. In Section 4 we study regularity for abstract Cauchy problems of the form

$$y'(t) + Ay(t) = f(y)(t), \qquad t > 0, y(0) = x, \tag{1.2}$$

where f: $C([0, T]; X) \mapsto L^1(0, T; X)$ is locally Lipschitzian and X is a Banach space, where A is an unbounded linear operator, generating a continuous semigroup $\{S(t)\}$ of bounded linear operators. The main tool is the formula $y(t) = S(t)x + \int_0^t S(t-s)f(y)(s)\,ds$. In Section 5 we transform our BVP into a problem of the type (1.2).

2. THE CASE $G = R = 0$

Now, the general solution of system (S) can be represented by explicit formulae of the D'Alembert type. However, for general G and R this is not possible (see, e.g., [4, p. 58]).

Let $R = G = 0$. We extend f_1 and f_2 on $[0, T] \times \mathbf{R}$ by

$$\begin{aligned} f_i(t, x) &= f_i(t, 2 - x), \qquad \text{for } 1 < x \le 2, i = 1, 2, \\ f_i(t, x) &= f_i(t, -x), \qquad \text{for } -1 \le x < 0, i = 1, 2, \end{aligned} \tag{2.1}$$

and so on. The *general solution* (u, v) of (S) is given by the following D'Alembert-type formulae, for each $t \in [0, T]$ and $x \in [0, 1]$,

$$\begin{aligned} u(t, x) &= \frac{1}{2}(\phi(x - t) + \psi(x + t) + h_1(t, x) + h_2(t, x)), \\ v(t, x) &= \frac{1}{2}(\phi(x - t) - \psi(x + t) + h_1(t, x) - h_2(t, x)), \end{aligned} \tag{2.2}$$

$$\begin{aligned} h_1(t, x) &= \int_{x-t}^{x} (f_1 + f_2)(t - x + s, s)\,ds, \\ h_2(t, x) &= \int_{x}^{x+1} (f_1 - f_2)(t + x - s, s)\,ds, \end{aligned} \tag{2.3}$$

where ϕ: $[-T, 1] \mapsto \mathbf{R}$ and ψ: $[0, 1 + T] \mapsto \mathbf{R}$ are arbitrary sufficiently smooth functions.

Assume (2.1)–(2.3). Then (IC) is equivalent to

$$\phi(x) = u_0(x) + v_0(x), \quad \psi(x) = u_0(x) - v_0(x), \qquad \text{for } 0 \le x \le 1. \tag{2.4}$$

Without any loss of generality we may assume that $T \leq 1$. Since β is maximal monotone, $(I+\beta)^{-1}$: $\mathbf{R}^2 \mapsto \mathbf{R}^2$ is a contraction. Thus (BC) is equivalent to

$$\begin{pmatrix} -\phi(-t) \\ \psi(1+t) \end{pmatrix} = \begin{pmatrix} h_1(t,0) - h_2(t,0) - \psi(t) \\ h_1(t,1) - h_2(t,1) + \phi(1-t) \end{pmatrix} - 2(I+\beta)^{-1} \begin{pmatrix} -\psi(t) - h_2(t,0) \\ \phi(1-t) + h_1(t,1) \end{pmatrix}, \tag{2.5}$$

for each $0 < t \leq T$.

If f_1 and f_2 are smooth enough, (2.1)–(2.5) yield the classical solution of our BVP. However, (2.1)–(2.5) make sense under weaker assumptions. Thus (u, v), given by (2.1)–(2.5), is called the *generalized solution* of our BVP whenever the integrals in (2.3) are well defined with respect to the Lebesgue measure.

PROPOSITION 2.1. *Assume that* $p \in [1,\infty]$, $f_1, f_2 \in L^\infty(0,T;L^p(0,1))$, $R = G = 0$, *and* $u_0, v_0 \in L^p(0,1)$. *Then our BVP has a unique generalized solution* $(u,v) \in L^\infty(0,T;L^p(0,1))^2$. *Moreover,* (u,v) *does not depend on how* f_1 *and* f_2 *are extended to* $L^\infty(0,T;L^p_{loc}(\mathbf{R}))$.

Proof. By (2.1), $f_1, f_2 \in L^\infty(0,T;L^p(-1,2))$. By [2, Sect. III.11.17], there are $\tilde{f}_1, \tilde{f}_2$: $[0,T]\times[-1,2] \mapsto \mathbf{R}$, measurable with respect to the product measure of $[0,T]\times[-1,2]$, and $\tilde{f}_1(t,\cdot) = f_1(t)$, $\tilde{f}_2(t,\cdot) = f_2(t)$, for a.e. $t \in |0,T|$. Moreover, $\tilde{f}_1$ and $\tilde{f}_2$ are uniquely determined except on a set whose product measure is zero. So we may study $(\tilde{f}_1, \tilde{f}_2)$ instead of (f_1, f_2).

Lemma 21.30 of [3] on the measurability of $f \circ \phi$ can easily be modified and proved for product measurable functions f: $\mathbf{R}^2 \mapsto \mathbf{R}$ and for Borel functions ϕ: $\mathbf{R}^2 \mapsto \mathbf{R}^2$. By this modified lemma, the functions $(t,s) \mapsto \tilde{f}_i(t \pm x \mp s, s)$, $i = 1,2$, are Lebesgue measurable. The functions

$$(t,s) \mapsto \begin{cases} 1, & \text{if } x - t \leq s \leq x, \\ 0, & \text{otherwise}, \end{cases} \qquad (t,s) \mapsto \begin{cases} 1, & \text{if } x \leq s \leq x + t, \\ 0, & \text{otherwise}, \end{cases}$$

are clearly Lebesgue measurable. Thus (2.3) can be rewritten as

$$h_1(t,x) = \int_{-1}^{2} g^x(t,s)\,ds, \qquad h_2(t,x) = \int_{-1}^{2} \tilde{g}^x(t,s)\,ds,$$

where $g^x, \tilde{g}^x$: $[0,T]\times[-1,2] \mapsto \mathbf{R}$ are some Lebesgue measurable functions. Since $f_1, f_2 \in L^\infty(0,T;L^p(-1,2))$, the iterated integrals of $|g_x|$ and $|\tilde{g}_x|$ over $[0,T]\times[-1,2]$ are finite. Thus by Fubini's theorem [3, Sect. 21.16–17], $t \mapsto h_1(t,x)$ and $t \mapsto h_2(t,x)$ are defined, for a.e. $t \in]0,T[$ and each $x \in [0,1]$, and they are Lebesgue measurable. Similarly,

$$x \mapsto h_1(t,x), \qquad x \mapsto h_2(t,x),$$
$$(x,s) \mapsto g^x(t,s), \qquad (x,s) \mapsto \tilde{g}^x(t,s)$$

are Lebesgue measurable for each $t \in [0, T]$. By Fubini's theorem,

$$\|h_1(t, \cdot)\|_{L^r(0,1)} + \|h_2(t, \cdot)\|_{L^r(0,1)} \\ \leq 2\|f_1\|_{L^\infty(0,T;L^p(0,1))} + 2\|f_2\|_{L^\infty(0,T;L^p(0,1))},$$

if $1 \leq r \leq p < \infty$. If $p = \infty$, we still have the above estimate for any $r \geq 1$. Thus in any case $h_1, h_2 \in L^\infty(0, T; L^p(0, 1))$. It is obvious that the functions ϕ and ψ, given by (2.4)–(2.5), are Lebesgue measurable. By [3, Sect. 21.31],

$$x \mapsto \phi(t - x), \qquad x \mapsto \psi(t + x),$$

$$(t, x) \mapsto \phi(t - x), \qquad (t, x) \mapsto \psi(t + x),$$

are Lebesgue measurable, too. By Fubini's theorem the last two functions belong to $L^\infty(0, T; L^p(0, 1))$, as h_1, h_2 above. Thus $u, v \in L^\infty(0, T; L^p(0, 1))$.

Let $\hat{f}_1, \hat{f}_2 \in L^\infty(0, T; L^p(-1, 2))$ be other extensions of f_1 and f_2. Denote by $(\hat{u}, \hat{v})$ the corresponding functions given by (2.2)–(2.4). By a direct calculation $(u, v) = (\hat{u}, \hat{v})$. Hence the generalized solution is also unique. ∎

PROPOSITION 2.2. *Assume the conditions of Proposition* 2.1 *and, in addition, that* $f_1, f_2 \in C([0, T]; L^p(0, 1))$. *Then our BVP has a unique generalized solution* $(u, v) \in C([0, T]; L^p(0, 1))^2$.

Proof. From the proof of Proposition 2.1 we see that $\phi \in L^p(-T, 1)$ and $\psi \in L^p(0, 1 + T)$. Since the set of continuous functions is dense in L^p, we obtain that $(t, x) \mapsto \phi(x - t)$, $(t, x) \mapsto \psi(x + t)$ belong to $C([0, T]; L^p(0, 1))$. By Fubini's theorem we conclude from $f_1, f_2 \in C([0, T]; L^p(0, 1))$ that

$$\limsup_{\epsilon \to 0} \int_0^1 |h_i(t + \epsilon, x) - h_i(t, x)|^p \, dx = 0, \qquad i = 1, 2.$$

Thus $u, v \in C([0, T]; L^p(0, 1))$. ∎

PROPOSITION 2.3. *Assume the conditions of Proposition* 2.1 *with* $p = 2$. *Then the weak solution of our BVP is its unique generalized solution.*

Proof. See Theorem 1.2. The mapping from $L^\infty(0, T; L^2(0, 1))^2$ into $C([0, T]; L^2(0, 1))^2$, $(f_1, f_2) \mapsto (u, v)$ is continuous by (2.1)–(2.5). Since the weak solution is the limit of classical solutions, given by D'Alembert formulae, the weak and generalized solutions coincide. ∎

PROPOSITION 2.4. *Assume that* $R = G = 0$, $f_1, f_2 \in C^1([0, T]; C[0, 1])$, $\beta = (\beta_1, \beta_2) \in C^1(\mathbf{R}^2)^2$, *and that* $u_0, v_0 \in C^1[0, 1]$ *satisfy* (1.1) *and the first-order compatibility conditions*

$$\begin{pmatrix} -f_1(0, 0) + v_0'(0) \\ f_1(0, 1) - v_0'(1) \end{pmatrix} = \beta'(v_0(0), v_0(1)) \begin{pmatrix} f_2(0, 0) - u_0'(0) \\ f_2(0, 1) - u_0'(1) \end{pmatrix}, \tag{2.6}$$

where $\beta'(v_0(0), v_0(1))\colon \mathbf{R}^2 \mapsto \mathbf{R}^2$ *is the Fréchet differential of* β. *Then the solution of our BVP belongs to* $C^1(\overline{D_T})^2$, *where* $D_T =]0, T[\times]0, 1[$.

Proof. Clearly, the extensions of f_1 and f_2 belong to $C^1([0, T]; C(\mathbf{R}))$. The functions ϕ and ψ are determined uniquely by (2.4)–(2.5). By (1.1) and (2.6), $\phi \in C^1[-T, 1]$ and $\psi \in C^1[0, 1+T]$. So, according to (2.2)–(2.3), $u, v \in C^1([0, T]; C[0, 1])$. Since (u, v) satisfies (S), we have $(u, v) \in C^1(\overline{D_T})^2$. ∎

PROPOSITION 2.5. *Let* $R = G = 0$, $f_1, f_2 \in C(\overline{D_T})$, *and* $u_0, v_0 \in C[0, 1]$ *satisfy* (1.1). *Then, our BVP has a unique generalized solution* $(u, v) \in C(\overline{D_T})^2$.

Proof. We just modify the proof of Proposition 2.4. ∎

Remark 2.1. Clearly, (u, v) given by Proposition 2.5 satisfies (BC) for $0 \le t \le T$. It is stronger than the weak solution given by Theorem 1.2.

Remark 2.2. Propositions 2.2 and 2.5 are valid for a t-dependent β.

Indeed, for Proposition 2.2 the measurability of $t \mapsto (I + \beta(t))^{-1}\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^2$ is enough; for Proposition 2.5 it suffices that these functions are continuous, and in (1.1) β is replaced by $\beta(0)$.

3. THE GENERAL CASE

We call (u, v) a *generalized solution* of our BVP if it satisfies (2.1)–(2.5), where (f_1, f_2) is replaced by $(f_1 - Ru, f_2 - Gv)$.

THEOREM 3.1. *Assume that* β, f_1, f_2, u_0, *and* v_0 *are as in Proposition* 2.5. *Then, our BVP has a unique generalized solution* $(u, v) \in C(\overline{D_T})^2$.

Proof. Consider the Banach space $Z_L = C(\overline{D_T})^2$ with the norm

$$\|(y, z)\|_L = \sup_{(t,x)\in\overline{D_T}} e^{-Lt}\|(y(t, x), z(t, x))\|_{\mathbf{R}^2}, \tag{3.1}$$

where $L > 0$ will be chosen later. Let $T < 1$, $K = \sqrt{2}\max(R, G)$, $\alpha = (\alpha_1, \alpha_2) \in Z_L$, $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) \in Z_L$, and h_1, h_2, ϕ, ψ, u, v and $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{\phi}$, $\tilde{\psi}$, $\tilde{u}$, $\tilde{v}$ be given by (2.1)–(2.5) with $(f_1 - R\alpha_1, f_2 - G\alpha_2)$, $(f_1 - R\tilde{\alpha}_1, f_2 - G\tilde{\alpha}_2)$ instead of (f_1, f_2), respectively. By Proposition 2.5 the mapping $P\colon Z_L \mapsto Z_L$, $P_\alpha = P(\alpha_1, \alpha_2) = (u, v)$, is well defined. By (2.1) and (2.3),

$$\begin{aligned}
&|h_1(t, x) - \tilde{h}_1(t, x)| + |h_2(t, x) - \tilde{h}_2(t, x)| \\
&\quad = \left| \int_{x-t}^{x} (R(\alpha_1 - \tilde{\alpha}_1) + G(\alpha_2 - \tilde{\alpha}_2))(t - x + s, s)\, ds \right| \\
&\qquad + \left| \int_{x}^{x+t} (R(\alpha_1 - \tilde{\alpha}_1) - G(\alpha_2 - \tilde{\alpha}_2))(t + x - s, s)\, ds \right|
\end{aligned}$$

$$\leq K \int_0^t (\|(\alpha - \tilde{\alpha})(\tau, \tau - t + x)\|_{\mathbf{R}^2} + \|(\alpha - \tilde{\alpha})(\tau, t + x - \tau)\|_{\mathbf{R}^2})\, d\tau$$
$$\leq 2K \int_0^t e^{L\tau} \|\alpha - \tilde{\alpha}\|_L d\tau \leq \frac{2K}{L} e^{Lt} \|\alpha - \tilde{\alpha}\|_L,$$

for any $(t, x) \in \overline{D_T}$. By (2.4), $\phi(x) = \tilde{\phi}(x)$ and $\psi(x) = \tilde{\psi}(x)$ on [0, 1]. Thus, by (2.5),

$$\left\| \begin{pmatrix} -\phi(-t) \\ \psi(1+t) \end{pmatrix} - \begin{pmatrix} -\tilde{\phi}(-t) \\ \tilde{\psi}(1+t) \end{pmatrix} \right\|_{\mathbf{R}^2}$$
$$\leq \left\| \begin{pmatrix} h_1(t,0) - \tilde{h}_1(t,0) \\ -h_2(t,1) + \tilde{h}_2(t,1) \end{pmatrix} \right\|_{\mathbf{R}^2} + 3 \left\| \begin{pmatrix} h_1(t,0) - \tilde{h}_1(t,0) \\ h_1(t,1) - \tilde{h}_1(t,1) \end{pmatrix} \right\|_{\mathbf{R}^2}$$
$$\leq \frac{16K}{L} e^{Lt} \|\alpha - \tilde{\alpha}\|_L,$$

for any $t \in]0, T]$. Hence,

$$|\phi(x-t) - \tilde{\phi}(x-t)| \leq \frac{16K}{L} e^{L(t-x)} \|\alpha - \tilde{\alpha}\|_L,$$
$$|\psi(x+t) - \tilde{\psi}(x+t)| \leq \frac{16K}{L} e^{L(t+x-1)} \|\alpha - \tilde{\alpha}\|_L,$$

for any $(t, x) \in \overline{D_T}$. By (2.2),

$$\|(u, v) - (\tilde{u}, \tilde{v})\|_L \leq \frac{36K}{L} \|\alpha - \tilde{\alpha}\|_L,$$

i.e., by choosing $L > 36K$, P: $Z_L \mapsto Z_L$ becomes a strict contraction. By Banach's fixed point theorem, P has a unique fixed point $\alpha \in Z_L$. This α is the desired unique generalized solution. ∎

Remark 3.1. By a similar approach, Propositions 2.1, 2.2, and 2.3 can be extended for general R and G.

THEOREM 3.2. *Assume that* β, f_1, f_2, u_0, *and* v_0 *are as in Proposition* 2.4, *except for* (2.6), *which is replaced by*

$$\begin{pmatrix} -f_1(0,0) + Ru_0(0) + v_0'(0) \\ f_1(0,1) - Ru_0(1) - v_0'(1) \end{pmatrix}$$
$$= \beta' \begin{pmatrix} v_0(0) \\ v_0(1) \end{pmatrix} \begin{pmatrix} f_2(0,0) - Gv_0(0) - u_0'(0) \\ f_2(0,1) - Gv_0(1) - u_0'(1) \end{pmatrix}. \tag{3.2}$$

Then our BVP has a unique classical solution $(u, v) \in C^1(\overline{D_T})^2$.

Proof. By Theorem 3.1, our BVP has a unique generalized solution $(u, v) \in C(\overline{D_T})^2$. Theorem 3.1 can be extended for the linear time-dependent case where monotone $\beta(t) \in L(\mathbf{R}^2, \mathbf{R}^2)$ are continuous with respect to t. So there is a unique generalized solution $(\tilde{u}, \tilde{v}) \in C(\overline{D_T})^2$ of the problem

$$\begin{aligned} \tilde{u}_t(t,x) + \tilde{v}_x(t,x) &= f_{1,t}(t,x) - R\tilde{u}(t,x), && \text{for } (t,x) \in D_T, \\ \tilde{v}_t(t,x) + \tilde{u}_x(t,x) &= f_{2,t}(t,x) - G\tilde{v}(t,x), && \text{for } (t,x) \in D_T, \end{aligned} \tag{3.3}$$

$$\begin{pmatrix} -\tilde{u}(t,0) \\ \tilde{u}(t,1) \end{pmatrix} \in \beta' \begin{pmatrix} v(t,0) \\ v(t,1) \end{pmatrix} \begin{pmatrix} \tilde{v}(t,0) \\ \tilde{v}(t,1) \end{pmatrix}, \qquad \text{for } 0 < t < T, \tag{3.4}$$

$$\begin{aligned} \tilde{u}(0,x) &= f_1(0,x) - Ru_0(x) - v_0'(x), && \text{for } 0 < x < 1, \\ \tilde{v}(0,x) &= f_2(0,x) - Gv_0(x) - u_0'(x), && \text{for } 0 < x < 1. \end{aligned} \tag{3.5}$$

On the other hand, (u, v) satisfies (2.1)–(2.5) with $(f_1 - Ru, f_2 - Gv)$ instead of (f_1, f_2). By differentiating this system with respect to t, we see that $(u_t, v_t) \in L^\infty(0, T; L^2(0,1))^2$ (see Theorem 1.1), and it is the generalized solution of (3.3)–(3.5). Since the generalized solution in $L^\infty(0, T; L^2(0,1))^2$ is unique, $(u_t, v_t) = (\tilde{u}, \tilde{v}) \in C(\overline{D_T})^2$. Thus by (S), $(u, v) \in C^1(\overline{D_T})^2$. ∎

Remark 3.2. Higher regularity can be shown under smoother data and higher order compatibility conditions (see [1]).

4. PERTURBED LINEAR SEMIGROUPS

Let $T > 0$, let X be a Banach space, let $A\colon D(A) \subset X \mapsto X$, and let f be a mapping from $C([0,T]; X)$ into $L^1(0, T; X)$, and consider the equation

$$u'(t) + Au(t) = f(u)(t), \qquad \text{for a.e. } t \in \,]0, T[. \tag{4.1}$$

A function $u \in W^{1,1}(0, T; X)$ is called a *strong solution* of (4.1) if $u(t) \in D(A)$ and Eq. (4.1) is satisfied almost everywhere on $]0, T[$. A *weak solution* of (4.1) is a function $u \in C([0,T]; X)$ such that there exist $f_n\colon C([0,T]; X) \mapsto L^1(0, T; X)$ and strong solutions $u_n \in C([0,T]; X)$ of

$$u_n'(t) + Au_n(t) = f_n(u_n)(t), \qquad \text{for a.e. } t \in \,]0, T[, n = 1, 2, \ldots, \tag{4.2}$$

satisfying

$$\lim_{n\to\infty} \|u_n - u\|_{C([0,T];X)} = \lim_{n\to\infty} \|f_n(u_n) - f(u)\|_{L^1(0,T;X)} = 0. \tag{4.3}$$

We impose the following conditions:

(H_A) The operator A is linear and $-A$ generates a continuous semigroup $\{S(t) \mid t \geq 0\}$ of bounded linear operators on X.

(H_f) There are u_0 in X and η in $L^1(0,T)$. If $\alpha \in Y_0 = \{\gamma \in C([0,T];X) \mid \gamma(0) = u_0\}$, then $f(\alpha) \in L^1(0,T;X)$ and

$$\|f(\alpha)(t)\|_X \leq \eta(t)\big(1 + \|\alpha\|_{L^\infty(0,t;X)}\big), \qquad \text{for a.e. } t \in]0,T[. \tag{4.4}$$

For each $K > 0$, there is $\eta_K \in L^1(0,T)$ such that

$$\|f(\alpha)(t) - f(\beta)(t)\|_X \leq \eta_K(t)\|\alpha - \beta\|_{L^\infty(0,t;X)}, \tag{4.5}$$

for a.e. $t \in]0,T[$, whenever $\alpha, \beta \in \{\gamma \in Y_0 \mid \|\gamma\|_{L^\infty(0,T;X)} \leq K\}$.

(H'_f) There exist u_0 in $D(A)$ and η in $L^1(0,T)$ with the following properties. For each $\alpha \in Y_1$, (4.4) is satisfied and $f(\alpha) \in W^{1,1}(0,T;X)$;

$$Y_1 = \{\gamma \in C^1([0,T];X) \mid \gamma(0) = u_0, \gamma'(0) = f(u_0)(0) - Au_0\}.$$

For each $K, L > K_1 = \|u_0\|_X + \|f(u_0)(0) - Au_0\|_X$, there are $\eta_K, \tilde{\eta}_L \in L^1(0,1)$ such that, for a.e. $t \in]0,T[$ and for each $\alpha, \beta, \gamma \in Y_1$,

$$\|f(\gamma)'(t)\|_X \leq \eta_K(t)(1 + \|\gamma'\|_{L^\infty(0,t;X)}), \tag{4.6}$$

$$\sum_{i=0}^{1} \|f(\alpha)^{(i)}(t) - f(\beta)^{(i)}(t)\|_X \leq \tilde{\eta}_L(t)\|\alpha - \beta\|_{W^{1,\infty}(0,t;X)}, \tag{4.7}$$

whenever $\|\gamma\|_{L^\infty(0,T;X)} \leq K$, $\|\alpha\|_{W^{1,\infty}(0,T;X)} \leq L$, $\|\beta\|_{W^{1,\infty}(0,T;X)} \leq L$.

(H''_f) There exist u_0 in $D(A)$ and η in $L^1(0,1)$ such that $f(u_0)'(0)$ exists and $f(u_0)(0) - Au_0 \in D(A)$. For each $\alpha \in Y_2$, (4.4) is satisfied and $f(\alpha) \in W^{2,1}(0,T;X)$;

$$Y_2 = \{\gamma \in C^2([0,T];X) \cap Y_1 \mid \gamma''(0) = f(u_0)'(0) - A(f(u_0)(0) - Au_0)\}.$$

Moreover, for each $K > K_1$ there exists $\eta_K \in L^1(0,T)$ satisfying (4.6), whenever $\gamma \in Y_2$ and $\|\gamma\|_{L^\infty(0,T;X)} \leq K$. For each $L, N > K_1$ there are $\tilde{\eta}_L, \hat{\eta}_N \in L^1(0,T)$ such that for a.e. $t \in]0,T[$,

$$\|f(\gamma)''(t)\|_X \leq \tilde{\eta}_L(t)(1 + \|\gamma''\|_{L^\infty(0,T;X)}), \qquad \text{for } \gamma \in Y_2, \tag{4.8}$$

$$\sum_{i=0}^{2} \|f(\alpha)^{(i)}(t) - f(\beta)^{(i)}(t)\|_X \leq \hat{\eta}_N(t)\|\alpha - \beta\|_{W^{2,\infty}(0,t;X)}, \tag{4.9}$$

whenever $\|\gamma'\|_{L^\infty(0,t;X)} \leq L$ and $\alpha, \beta \in \{\gamma \in Y_2 \mid \|\gamma\|_{W^{2,\infty}(0,T;X)} \leq N\}$.

THEOREM 4.1. *Assume* (H_A) *and* (H_f). *Then Equation* (4.1) *has a unique weak solution* $u \in C([0,T];X)$ *satisfying* $u(0) = u_0$.

THEOREM 4.2. *Assume* (H_A), (H_f), $R(f) \subset W^{1,1}(0,T;X)$, *and* $u_0 \in D(A)$. *Then there exists a unique strong solution* u *in* $W^{1,1}(0,T;X)$ *of Equation* (4.1) *satisfying* $u(0) = u_0$.

THEOREM 4.3. *Assume* (H_A) *and* (H_f'). *Then Equation* (4.1) *together with the initial condition* $u(0) = u_0$ *has a unique* (*classical*) *solution* $u \in C^1([0,T];X)$, *and* $Au \in C([0,T];X)$.

THEOREM 4.4. *Assume* (H_A) *and* (H_f''). *Then* (4.1) *with* $u(0) = u_0$ *has a unique* (*classical*) *solution* $u \in C^2([0,T];X)$, $R(u') \subset D(A)$, *and* $Au' = (Au)' \in C([0,T];X)$. *If, in addition*, $R(f(\alpha)) \subset D(A)$ *and* $Af(\alpha) \in C([0,T];X)$, *for any* $\alpha \in Y_2$, *then* $R(u) \subset D(A^2)$ *and* $A^2u \in C([0,T];X)$.

The assumption that f is locally Lipschitz can be partly replaced by stronger differentiability conditions. Indeed, by the idea of the proof of Theorem 3.2, we can prove, for example, the following theorem. Observe that one could similarly formulate higher order regularity results.

THEOREM 4.5. *Assume the conditions of Theorem* 4.3 *and, in addition, that there exists a mapping* f^* *from* $C^1([0,T];X) \times C([0,T];X)$ *into* $L^1(0,T;X)$ *such that* $f(\alpha) \in C^1([0,T];X)$, $f(\alpha)'(t) = f^*(\alpha,\alpha')(t)$, $0 \le t \le T$, *and the couple* $(f(u_0)(0) - Au_0, f^*(\alpha,\cdot))$ *satisfies* (H_f'), *for any given* $\alpha \in C^1([0,T];X)$ *such that* $\alpha(0) = u_0$ *and* $\alpha'(0) = f(u_0) - Au_0$. *Then* (4.1) *has a unique* (*classical*) *solution* $u \in C^2([0,T];X)$ *satisfying* $u(0) = u_0$, $R(u') \subset D(A)$, *and* $Au' = (Au)' \in C([0,T];X)$.

Proof of Theorem 4.1. By [6, p. 20], $\overline{D(A)} = X$. Thus there are $u_{0n} \in D(A)$, $n = 1,2,\ldots$, converging in X toward u_0. Denote by χ_B the characteristic function of the set B, let $\alpha \in Y_0$, and consider the problems

$$u_n'(t) + Au_n(t) = f_n(\alpha)(t), \qquad \text{for a.e. } t \in]0,T[,\ u_n(0) = u_{0n}, \tag{4.10}$$

$$f_n(\alpha)(t) = \int_{\mathbf{R}} \rho_{1/n}(t-s)\chi_{[0,T]}(s) f(\alpha)(s)\,ds, \tag{4.11}$$

where $\rho_{1/n} \in C_0^\infty(\mathbf{R})$ is the usual mollifier satisfying $\int_{\mathbf{R}} \rho_{1/n}(s)\,ds = 1$, $\rho_{1/n} \ge 0$, and $\operatorname{supp} \rho_{1/n} \subset [-1/n, 1/n]$. Since $f_n(\alpha) \in C^\infty([0,T];X)$, then by [6, p. 109], (4.10) has unique strong solutions $u_n \in W^{1,1}(0,T;X)$ and

$$u_n(t) = S(t)u_{0n} + \int_0^t S(t-s) f_n(\alpha)(s)\,ds, \qquad \text{for each } t \in [0,T]. \tag{4.12}$$

By [6, p. 4], there are constants $M, \omega \ge 0$ satisfying

$$\|S(t)\|_{L(X;X)} \le Me^{\omega t}, \qquad \text{for each } t \ge 0. \tag{4.13}$$

By (4.12) and (4.13), for each $t \in [0,T]$ and $m,n = 1,2,\ldots$,

$$\|u_n(t) - u_m(t)\|_X \le Me^{\omega T}\big(\|u_{0m} - u_{0n}\| + \|f_n(\alpha) - f_m(\alpha)\|_{L^1(0,T;X)}\big).$$

Since $f_n(\alpha) \to f(\alpha)$ in $L^1(0,T;X)$ as $n \to \infty$, then (u_n) converges toward some u_α in $C([0,T];X)$. Hence the equation

$$u_\alpha'(t) + Au_\alpha(t) = f(\alpha)(t), \qquad \text{for a.e. } t \in]0,T[,\ u_\alpha(0) = u_0, \tag{4.14}$$

has a weak solution $u_\alpha \in C([0,T];X)$ given by

$$u_\alpha(t) = S(t)u_0 + \int_0^t S(t-s)f(\alpha)(s)\,ds, \qquad \text{for each } t \in [0,T]. \tag{4.15}$$

Let $\tilde{u}_\alpha \in C([0,T];X)$ be another weak solution of (4.14). Then there are $\tilde{f}_n(\alpha) \in L^1(0,T;X)$ and strong solutions $\tilde{u}_n$ of (4.14) with $\tilde{f}_n(\alpha)$ instead of $f(\alpha)$ such that $\tilde{f}_n(\alpha) \to f(\alpha)$ in $L^1(0,T;X)$ and $\tilde{u}_n \to \tilde{u}_\alpha$ in $C([0,T];X)$, as $n \to \infty$. By (4.13) and by (4.12), for each $t \in [0,T]$,

$$\|u_\alpha(t) - \tilde{u}_\alpha(t)\|_X \le \limsup_{n\to\infty} Me^{\omega T}\|f_n(\alpha) - \tilde{f}_n(\alpha)\|_{L^1(0,T;X)} = 0.$$

Thus u_α is unique. So P: $Y_0 \mapsto Y_0$, $P\alpha = u_\alpha$, is well defined.

Let $t \in]0,T]$ and $\xi \in L^1(0,T)$ be positive. Define, for each $\alpha \in L^\infty(0,t;X)$,

$$\|\alpha\|_{\xi,t} = \operatorname*{ess\,sup}_{0\le s\le t} e^{-\int_0^t \xi(\sigma)d\sigma}\|\alpha(s)\|_X. \tag{4.16}$$

Using (4.15), (4.13), and $\xi = Me^{\omega T}\eta$, we obtain, for any $\alpha \in C([0,T];X)$,

$$\|P\alpha\|_{\xi,T} \le \left(1 - e^{-\int_0^T \xi(\sigma)d\sigma}\right)\|\alpha\|_{\xi,T} + e^{-\int_0^T \xi(\sigma)d\sigma}Me^{\omega T}\left(\|u_0\|_X + \|\eta\|_{L^2(0,T)}\right).$$

Denote $Z_L = \{\gamma \in Y_0 \mid \|\gamma\|_{\xi,T} \le L\}$. We choose $L > Me^{\omega T}\big(\|u_0\|_X + \|\eta\|_{L^1(0,T)}\big)$. Then P: $Z_L \mapsto Z_L$. Let $K = L\exp\int_0^T \xi(\sigma)d\sigma$. So $K\|\gamma\|_{\xi,T} \ge L\|\gamma\|_{L^\infty(0,T;X)}$. We use in Z_L the metrics induced by $\|\cdot\|_{\xi_L,T}$, $\xi_L = Me^{\omega T}\eta K$. Now, Z_L is a complete metric space. Using (4.15), (4.13), and (4.5), we calculate that

$$\|P\alpha - P\beta\|_{\xi_L,T} \le \left(1 - e^{-\int_0^T \xi_L(\sigma)d\sigma}\right)\|\alpha - \beta\|_{\xi_L,T}, \tag{4.17}$$

for each $\alpha, \beta \in Z_L$. Thus P has a unique fixed point $u \in Z_L$ which is the desired weak solution of (4.1). Indeed, if there were another weak solution $\tilde{u}$ of (4.1), we could choose L above so big that $\tilde{u} \in Z_L$.

Proof of Theorem 4.2. Let $\alpha \in Y_0$. By [6, p. 109], the problem (4.14) has a unique strong solution $u_\alpha \in W^{1,1}(0,T;X)$, satisfying (4.15). Hence the above reasoning gives the existence of a unique strong solution of (4.1).

Proof of Theorem 4.3. Let $\alpha \in Y_1$, $n = 1, 2, \ldots$, and let u_n be the strong solution of (4.10) with $u_n(0) = u_0$. Then, as in (4.12),

$$u_n(t) = S(t)u_0 + \int_0^t S(t-s)f_n(\alpha)(s)\,ds, \qquad \text{for each } t \in [0,T]. \tag{4.18}$$

Since $f_n(\alpha)' \in C([0,T];X)$ and $u_0 \in D(A)$, we can differentiate (4.18). Thus

$$u_n'(t) = S(t)(f(\alpha)(0) - Au_0) + \int_0^t S(s)f_n(\alpha)'(t-s)\,ds, \tag{4.19}$$

for each $t \in [0, T]$; thus $u_n \in C^1([0, T]; X)$. Since $f_n(\alpha)$ tends to f in $W^{1,1}(0, T; X)$ and (4.13) holds, there is $u_\alpha \in C^1([0, T]; X)$, satisfying for each $t \in [0, T]$,

$$u_n \to u_\alpha \quad \text{in } C^1([0, T]; X), \qquad \text{as } n \to \infty, \tag{4.20}$$

$$u_\alpha(t) = S(t)u_0 + \int_0^t S(t-s)f(\alpha)(s)\, ds, \tag{4.21}$$

$$u'_\alpha(t) = S(t)(f(\alpha)(0) - Au_0) + \int_0^t S(t-s)f(\alpha)'(s)\, ds. \tag{4.22}$$

Since A is closed, (4.20) implies that u_α is a classical solution of (4.14). Since the weak solution of (4.14) is unique, so is the classical one. Thus the mapping P: $Y_1 \mapsto Y_1$, $P\alpha = u_\alpha$, is well defined, since by (4.5), by the continuity of $f(\alpha)$ and of α', $f(\alpha)(0) = f(u_0)(0)$. Let $\xi \in L^1(0, T)$ be positive and, for any $\beta \in Y_1$,

$$\|\beta\|'_{\xi,T} = \operatorname*{ess\,sup}_{0\le s\le T} e^{-\int_0^s \xi(\sigma)d\sigma}\big(\|\beta(s)\|_X + \|\beta'(s)\|_X\big). \tag{4.23}$$

A calculation using (4.20), (4.21), (4.4), and (4.6) reveals that we can choose first K and then L such that P: $\widehat{Z}_L \mapsto \widehat{Z}_L$, where $\widehat{Z}_L = \{\gamma \in Y_1 \mid \|\gamma\|_{\xi,T} \le L\}$ and $\xi = Me^{\omega T}(\eta + \eta K)$. The set $\widehat{Z}_L$ is a complete metric space if its metrics is induced by the norm $\|\cdot\|'_{\tilde{\xi},T}$, where $\tilde{\xi} = Me^{\omega T}\tilde{\eta}_{L'}$, and $L' = L \exp \int_0^T \xi(\sigma)d\sigma$.

Let $\alpha, \beta \in \widehat{Z}_L$. Then, by (4.21), (4.22), (4.13), and (4.7),

$$\|P\alpha - P\beta\|'_{\tilde{\xi},T} \le \left(1 - e^{-\int_0^T \tilde{\xi}(\sigma)d\sigma}\right)\|\alpha - \beta\|'_{\tilde{\xi},T}. \tag{4.24}$$

Thus P has a unique fixed point $u \in \widehat{Z}_L$. The required solution is found.

Proof of Theorem 4.4. Let us recall [6, pp. 20, 9] that the resolvent of A, $R(\lambda\colon A) = (\lambda I + A)^{-1}$, satisfies, for each $\lambda > \omega$ and $x \in X$,

$$\|R(\lambda\colon A)\|_{L(X;X)} \le \frac{M}{\lambda - \omega} \quad \text{and} \quad \lim_{\lambda\to\infty} \lambda R(\lambda\colon A)x = x, \tag{4.25}$$

with $M, \omega \ge 0$ from (4.13). Let $n > \omega$, $\alpha \in Y_2$, and $\tilde{f}_n(\alpha) = nR(n\colon A)f_{n^2}(\alpha)$, where $f_n(\alpha)$ are given by (4.11). Now $\tilde{f}_n(\alpha) \in C^\infty([0, T]; X)$. By the properties of the convolution approximation, for each $t \in [0, T]$ and $n = 1, 2, \ldots,$

$$\|f_n(\alpha)(t) - f(\alpha)(t)\|_X \le \frac{1}{n}\|f(\alpha)'\|_{L^\infty(0,T;X)}, \tag{4.26}$$

$$\lim_{n\to\infty} \|f_n(\alpha) - f(\alpha)\|_{W^{2,1}(0,T;X)} = 0. \tag{4.27}$$

By [6, p. 107], the problem

$$\begin{aligned} u_n'(t) + Au_n(t) &= \tilde{f}_n(\alpha)(t), \qquad \text{for a.e. } t \in]0, T[, \\ u_n(0) &= nR(n\colon A)u_0, \end{aligned} \tag{4.28}$$

has a unique strong solution u_n given by

$$u_n(t) = S(t)nR(n\colon A)u_0 + \int_0^t S(s)\tilde{f}_n(\alpha)(t-s)\,ds, \tag{4.29}$$

on $[0, T]$. Since A and $R(n\colon A)$ commute and $\tilde{f}_n(\alpha)' \in C([0, T]; X)$, we have

$$\begin{aligned} u_n'(t) &= S(t)nR(n\colon A)(f_{n^2}(\alpha)(0) - Au_0) \\ &\quad + \int_0^t S(s)\tilde{f}_n(\alpha)'(t-s)\,ds, \qquad \text{for each } t \in [0, T]. \end{aligned} \tag{4.30}$$

We differentiate again, since $\tilde{f}_n(\alpha)'' \in C([0, T]; X)$. Thus

$$\begin{aligned} u_n''(t) &= -S(t)AnR(n\colon A)(f_{n^2}(\alpha)(0) - Au_0) + S(t)\tilde{f}_n(\alpha)'(0) \\ &\quad + \int_0^t S(s)\tilde{f}_n(\alpha)''(t-s)\,ds, \qquad \text{for each } t \in [0, T]. \end{aligned} \tag{4.31}$$

Calculations using (4.13), (4.26)–(4.27) reveal that the right-hand sides of (4.29)–(4.31) converge uniformly. So there exists $u \in C^2([0, T]; X)$ such that, for each $t \in [0, T]$,

$$u(t) = S(t)u_0 + \int_0^t S(s)f(\alpha)(t-s)\,ds, \tag{4.32}$$

$$u'(t) = S(t)(f(u_0)(0) - Au_0) + \int_0^t S(s)f(\alpha)'(t-s)\,ds, \tag{4.33}$$

$$\begin{aligned} u''(t) &= S(t)A(Au_0 - f(u_0)(0)) + S(t)f(\alpha)'(0) \\ &\quad + \int_0^t S(s)f(\alpha)''(t-s)\,ds, \end{aligned} \tag{4.34}$$

$$u_n \to u \quad \text{in } C^2([0, T]; X), \qquad \text{as } n \to \infty. \tag{4.35}$$

Since A is closed, (4.28), (4.35), and (4.26) imply

$$u(t) \in D(A) \quad \text{and} \quad u'(t) + Au(t) = f(\alpha)(t), \qquad \text{for each } t \in [0, T]. \tag{4.36}$$

Using the linearity of A and (4.28), we obtain

$$\frac{u_n'(t+h) - u_n'(t)}{h} + A\frac{u_n(t+h) - u_n(t)}{h} = \frac{\tilde{f}_n(\alpha)(t+h) - \tilde{f}_n(\alpha)(t)}{h},$$

for each $t \in [0, T]$ and $h \in [-t, T-t]\setminus\{0\}$. As $h \to 0$ and $n \to \infty$, successively, by the closedness of A, by u_n, $\tilde{f}_n(\alpha) \in C^2([0,T]; X)$, (4.35), and (4.27),

$$u'(t) \in D(A) \quad \text{and} \quad u''(t) + Au'(t) = f(\alpha)'(t), \tag{4.37}$$

for each $t \in [0, T]$. Since $f(\alpha)'$ and u'' are continuous, so is Au'. Since A is closed, $Au' = (Au)'$. Let $t \in [0, T]$. If $f(\alpha)(t) \in D(A)$, then by (4.36)–(4.37), $Au(t) = f(\alpha)(t) - u'(t) \in D(A)$. Thus $u(t) \in D(A^2)$. If, in addition, $Af(\alpha) \in C([0,T]; X)$, then by (4.36) $A^2 u = Af(\alpha) - Au' \in C([0,T]; X)$.

The mapping P: $Y_2 \mapsto Y_2$, $P\alpha = u_\alpha$, the unique solution of (4.36) with $u(0) = u_0$, is again well defined. Let $\xi \in L^1(0,T)$ be positive and define

$$\|\beta\|^*_{\xi,T} = \operatorname*{ess\,sup}_{0\le s\le t} e^{-\int_0^s \xi(\sigma)d\sigma}\big(\|\beta(s)\|_X + \|\beta'(s)\|_X + \|\beta''(s)\|_X\big),$$

for each $\beta \in W^{2,\infty}(0,T;X)$. A calculation, using (4.32)–(4.34), (4.4), (4.6), and (4.8), shows that we can choose first K, then L, and finally N such that P: $Z_N^* \mapsto Z_N^*$, where $Z_N^* = \{\gamma \in Y_2 \mid \|\gamma\|^*_{\xi,T} \le N\}$ and $\xi = Me^{\omega T}(\eta + \eta_K + \tilde{\eta}_L)$. We choose $N' = N \exp \int_0^T \xi(\sigma)d\sigma$ and use in Z_N^* the metrics, induced by $\|\cdot\|^*_{\xi^*,T}$, where $\xi^* = Me^{\omega T}\hat{\eta}_{N'}$. Then by (4.8), (4.13), and (4.32)–(4.34),

$$\|P\alpha - P\beta\|^*_{\xi^*,T} \le \Big(1 - e^{-\int_0^T \hat{\xi}(\sigma)d\sigma}\Big)\|\alpha - \beta\|^*_{\xi^*,T}, \qquad \text{for each } \alpha, \beta \in Z_N^*.$$

Hence P has a unique fixed point $u \in Y_N^*$. Theorem 4.4 is proved.

Proof of Theorem 4.5. By Theorem 4.3, Eq. (4.1) and $u(0) = 0$ have a unique solution $u \in C^1([0,T]; X)$, given by (4.32) with $\alpha = u$. By Theorem 4.3 there exists a unique $v \in C^1([0,T]; X)$, satisfying

$$v'(t) + Av(t) = f^*(u, v)(t), \qquad 0 < t < T, v(0) = f(u_0)(0) - Au_0. \tag{a}$$

On the other hand, by differentiating u,

$$u'(t) = S(t)\big(f(u_0)(0) - Au_0\big) + \int_0^t S(s) f^*(u, u')(t-s)\, ds,$$

for each $t \in [0, T]$. Thus u' is the mild solution of (a). Since the mild solution is unique, $u' = v$. Hence $u \in C^2([0,T]; X)$, $Ru' \subset D(A)$, and $Au' \in C([0,T]; X)$. By the closedness of A, $(Au)' = Au'$.

5. SEMIGROUP APPROACH

Let $0 < T < 1$. We return to the study of (S), (BC), (IC). We shall transform it to a boundary value problem with homogeneous boundary conditions, namely,

$$\begin{aligned} \hat{u}_t(t,x) + \hat{v}_x(t,x) &= \hat{f}_1(\hat{u},\hat{v})(t,x), \qquad \text{for } (t,x) \in D_T, \\ \hat{v}_t(t,x) + \hat{u}_x(t,x) &= \hat{f}_2(\hat{u},\hat{v})(t,x), \qquad \text{for } (t,x) \in D_T, \end{aligned} \tag{5.1}$$

$$\hat{u}(t,0) = \hat{u}(t,1) = 0, \qquad \text{for } 0 < t < T, \tag{5.2}$$

$$\hat{u}(0,x) = \hat{u}_0(x), \quad \hat{v}(0,x) = \hat{v}_0(x), \qquad \text{for } 0 < x < 1. \tag{5.3}$$

For each given $\hat{u}, \hat{v} \in C([0,T]; C[0,1])$, we define a, c, d: $[-T, 1] \mapsto \mathbf{R}$, b: $[0, T+1] \mapsto \mathbf{R}$, and $u, v, \hat{f}_1(\hat{u},\hat{v}), \hat{f}_2(\hat{u},\hat{v})$: $\overline{D_T} \mapsto \mathbf{R}$. First we write

$$\begin{aligned} u(t,x) &= e^{-Rt}\big(\hat{u}(t,x) + a(x-t) + b(x+t) \\ &\qquad + x(1-x)^2 c(x-t) - x^2(1-x) d(x-t)\big), \\ v(t,x) &= e^{-Rt}\big(\hat{v}(t,x) + a(x-t) - b(x+t) \\ &\qquad + x(1-x)^2 c(x-t) - x^2(1-x) d(x-t)\big), \end{aligned} \tag{5.4}$$

$$\begin{aligned} \hat{f}_1(\hat{u},\hat{v})(t,x) &= e^{Rt} f_1(t,x) - (1-x)(1-3x)c(x-t) \\ &\qquad - x(3x-2)d(x-t), \\ \hat{f}_2(\hat{u},\hat{v})(t,x) &= e^{-Rt}\big(f_2(t,x) - (G-R)v(t,x)\big) \\ &\qquad - (1-x)(1-3x)c(x-t) - x(3x-2)d(x-t), \end{aligned} \tag{5.5}$$

$$c(-t) = e^{Rt} f_1(t,0), \qquad d(1-t) = e^{Rt} f_1(t,1), \tag{5.6}$$

for each $(t,x) \in \overline{D_T}$. Next, c and d are continued smoothly to $[-T, 1]$. We set $\tilde{\beta}(t) = I - (I + \mathrm{e}^{Rt}\beta\mathrm{e}^{-Rt})^{-1}$ and write, for each $0 \le x \le 1$ and $0 < t \le T$,

$$\begin{aligned} a(x) &= \frac{1}{2}(u_0(x) + v_0(x)) - x(1-x)^2 c(x) + x^2(1-x) d(x), \\ b(x) &= \frac{1}{2}(u_0(x) - v_0(x)), \end{aligned} \tag{5.7}$$

$$\begin{pmatrix} -a(-t) \\ b(1+t) \end{pmatrix} = \tilde{\beta}(t) \begin{pmatrix} \hat{v}(t,0) - 2b(t) \\ \hat{v}(t,1) + 2a(1-t) \end{pmatrix} - \begin{pmatrix} -b(t) \\ a(1-t) \end{pmatrix}. \tag{5.8}$$

If $\hat{u}$ and $\hat{v}$ are C^1-functions, satisfying (5.1)–(5.2), and a, b, c, d are also C^1-functions, then u, v satisfy (S) and (BC). Indeed, (5.8) is equivalent to (BC) under (5.2) and (5.4). Moreover, by choosing $\hat{u}_0 = \hat{v}_0 = 0$, we obtain the solution of (S), (BC), (IC) from the solution of (5.1)–(5.3).

By Proposition 2.5, (5.1)–(5.3) with $\hat{f}_1 = \hat{f}_2 = 0$ and $T = \infty$ have a unique generalized solution $(\tilde{u}, \tilde{v}) \in C([0, \infty[\times[0, 1])^2$, if $\hat{u}_0(0) = \hat{u}_0(1) = 0$. Hence we define a Banach space

$$X = C_0[0, 1] \times C[0, 1], \qquad \|(y, z)\|_X = \|y\|_{C[0,1]} + \|z\|_{C[0,1]}. \tag{5.9}$$

Then $S(t)\colon X \mapsto X$, $S(t)(\hat{u}_0, \hat{v}_0) = (\tilde{u}(t), \tilde{v}(t))$ is well defined, for any $t \geq 0$.

LEMMA 5.1. *$\{S(t) \mid t \geq 0\}$ is a strongly continuous semigroup of bounded linear operators $X \mapsto X$ and is generated by $-A$, where $A(y, z) = (z', y')$,*

$$D(A) = \{(y, z) \in C^1[0, 1]^2 \mid y(0) = y(1) = z'(0) = z'(1) = 0\}. \tag{5.10}$$

Proof. Let us consider the Hilbert space $H = L^2(0, 1)^2$, and the operator $B \subset H \times H$, given by

$$\big((y_1, y_2), (z_1, z_2)\big)_H = (y_1, z_1)_{L^2(0,1)} + (y_2, z_2)_{L^2(0,1)},$$
$$D(B) = \{(y, z) \in H^1(0, 1)^2 \mid y(0) = y(1) = 0\},$$
$$B(y, z) = (z', y').$$

Since B is linear and maximal monotone, $-B$ generates a strongly continuous semigroup $\{T(t) \mid t \geq 0\}$ of linear contractions $H \mapsto H$. On the other hand, $T(\cdot)(\hat{u}_0, \hat{v}_0) \in C([0, \infty[; H)$, and it is the weak solution of (5.1)–(5.3) with $\hat{f}_1 = \hat{f}_2 = 0$ (see Theorem 1.2). By Proposition 2.3, it coincides with $(\tilde{u}, \tilde{v})$,

$$\|T(t)(\hat{u}_0, \hat{v}_0) - S(t)(\hat{u}_0, \hat{v}_0)\|_H = 0, \qquad \text{for each } t \geq 0. \tag{5.11}$$

So also $\{S(t) \mid t \geq 0\}$ is a semigroup of linear operators, since $S(\cdot)(y, z) \in C([0, \infty[\times[0, 1])$ whenever $(y, z) \in X$. Direct calculations with (2.2)–(2.5) using the Weierstrass Theorem and the Mean Value Theorem show that

$$\|S(t)\|_{L(X;X)} \leq 2, \qquad \text{for each } t \in [0, 1], \tag{5.12}$$

$$\lim_{t\to 0+} \|S(t)(y, z) - (y, z)\|_X = 0, \qquad \text{for each } (y, z) \in X, \tag{5.13}$$

$$\lim_{t\to 0+} \|(S(t)(y, z) - (y, z))/t + A(y, z)\|_X = 0, \tag{5.14}$$

for each $(y, z) \in D(A)$. By (5.12)–(5.13), $\{S(t) \mid t \geq 0\}$ is a strongly continuous semigroup of bounded linear operators. By (2.2)–(2.4),

$$D(A) = \left\{(y, z) \in X \,\middle|\, \lim_{t\to 0+} \frac{S(t)(y, z) - (y, z)}{t} \text{ exists in } X\right\}. \tag{5.15}$$

By (5.14), the generator of $\{S(t) \mid t \geq 0\}$ is $-A$. ∎

Remark 5.1. Since A is not accretive, the Lumer–Phillips Theorem could not be used. However, the range condition $R(\lambda I + A) = X$, for each $\lambda > 0$, is satisfied.

Indeed, for any $\lambda \neq 0$ and $g = (g_1, g_2) \in X$, the equation $(\lambda I + A)(u, v) = (g_1, g_2)$ is equivalent with the problem

$$\lambda u(x) + v'(x) = g_1(x), \quad \lambda v(x) + u'(x) = g_2(x), \qquad \text{for each } x \in [0, 1],$$
$$u, v \in C^1[0, 1], \qquad u(0) = u(1) = 0,$$

since $v'(0) = v'(1) = 0$, by $g_1 \in C_0[0, 1]$. By the method of variation of constants we can see that

$$u(x) = C_2 \sinh \lambda x + \int_0^x (-g_1(y) \sinh \lambda(x - y) + g_2(y) \cosh \lambda(x - y))\, dy,$$
$$v(x) = -C_2 \cosh \lambda x + \int_0^x (g_1(y) \cosh \lambda(x - y) - g_2(y) \sinh \lambda(x - y))\, dy,$$
$$C_2 = \frac{1}{\sinh \lambda} \int_0^1 (g_1(y) \sinh \lambda(1 - y) - g_2(y) \cosh \lambda(1 - y))\, dy.$$

Thus $R(\lambda I + A) = X$.

By choosing $u(x) = \sin \pi x$, $v(x) = 2 \cos \pi x$, and $\lambda = \frac{1}{4\pi}$,

$$\|(u, v)\|_X = \|u\|_{C[0,1]} + \|v\|_{C[0,1]} = 1 + 2 = 3,$$
$$\|(u, v) + \lambda A(u, v)\|_X = \|u + \lambda v'\|_{C[0,1]} + \|v + \lambda u'\|_{C[0,1]} = 3 - \lambda\pi.$$

Thus $\|(u, v)\|_X \leq \|(u, v) + \lambda A(u, v)\|_X$ is not satisfied, i.e., A is not accretive.

The proofs of Lemmas 5.2–5.4 are straightforward. Observe that the compatibility conditions are needed in proving a and b to be smooth at 0 and at 1, respectively.

LEMMA 5.2. *Assume that f_1 and f_2 are continuous. Let $u_0, v_0 \in C[0, 1]$ satisfy* (1.1). *Then $a, c, d \in C[-T, 1]$ and $b \in C[0, 1 + T]$, for each $\hat{u}, \hat{v} \in C([0, T]; X)$ such that $\hat{u}(0, \cdot) = \hat{v}(0, \cdot) = 0$. Moreover, $\hat{f} = (\hat{f}_1, \hat{f}_2)$ satisfies* $(\mathrm{H_f})$ *with zero as the initial value.*

LEMMA 5.3. *Assume that $f_1, f_2 \in C^1([0, T]; C[0, 1])$, $u_0, v_0 \in C^1[0, 1]$, and $\beta \in C^1(\mathbf{R}^2)^2$ satisfy the compatibility conditions* (1.1) *and* (3.2). *Then $a, c, d \in C^1[-T, 1]$ and $b \in C^1[0, 1 + T]$, for each $\hat{u}, \hat{v} \in C^1([0, T]; X)$ such that $\hat{u}(0, \cdot) = \hat{v}(0, \cdot) = 0$, $\hat{u}_t(0, \cdot) = \hat{f}_1(0, 0)(0, \cdot)$ and $\hat{v}_t(0, \cdot) = \hat{f}_2(0, 0)(0, \cdot)$. If, in addition, $\beta'(\cdot)\mathbf{x}$ is locally Lipschitzian, for each $\mathbf{x} \in \mathbf{R}^2$, then $\hat{f} = (\hat{f}_1, \hat{f}_2)$ satisfies* $(\mathrm{H'_f})$ *with zero as the initial value.*

LEMMA 5.4. *Let* $f_1, f_2 \in C^2([0,T]; C[0,1])$, $u_0, v_0 \in C^2[0,1]$, *and* $\beta \in C^2(\mathbf{R}^2)^2$ *satisfy* (1.1), (3.2), *and*

$$\begin{pmatrix} -f_{1t}(0,0) + f_{2x}(0,0) - Gv_0'(0) - u_0''(0) \\ f_{1t}(0,1) - f_{2x}(0,1) + Gv_0'(1) + u_0''(1) \end{pmatrix}$$
$$= \beta''\begin{pmatrix} v_0(0) \\ v_0(1) \end{pmatrix}\begin{pmatrix} f_2(0,0) - Gv_0(0) - u_0'(0) \\ f_2(0,1) - Gv_0(1) - u_0'(1) \end{pmatrix}$$
$$+ \beta'\begin{pmatrix} v_0(0) \\ v_0(1) \end{pmatrix}\begin{pmatrix} f_{2t}(0,0) - f_{1x}(0,0) - v_0''(0) \\ \quad + (R-G)(f_2(0,0) - Gv_0(0) - u_0'(0)) \\ f_{2t}(0,1) - f_{1x}(0,1) - v_0''(1) \\ \quad + (R-G)(f_2(0,1) - Gv_0(1) - u_0'(1)) \end{pmatrix},$$

where β'' *is the second-order differential of* β. *Then* $a, c, d \in C^2[-T, 1]$ *and* $b \in C^2[0, 1+T]$, *for each* $\hat{u}, \hat{v} \in C^2([0,T]; X)$ *such that*

$$\hat{u}(0,\cdot) = 0, \qquad \hat{u}_t(0,\cdot) = \hat{f}_1(0,0)(0,\cdot),$$
$$\hat{v}(0,\cdot) = 0, \qquad \hat{v}_t(0,\cdot) = \hat{f}_2(0,0)(0,\cdot),$$
$$\hat{u}_{tt}(0,\cdot) = \hat{f}_1(0,0)_t(0,\cdot) - \hat{f}_2(0,0)_x(0,\cdot),$$
$$\hat{v}_{tt}(0,\cdot) = \hat{f}_2(0,0)_t(0,\cdot) - \hat{f}_1(0,0)_x(0,\cdot).$$

If, in addition, $\beta'(\cdot)\mathbf{x}$ *and* $\beta''(\cdot)\mathbf{x}$ *are locally Lipschitzian, for each* $\mathbf{x} \in \mathbf{R}^2$, $f_1, f_2 \in C^1(\overline{D_T})$, *and*

$$2c(1) = f_{2x}(0,1) - (G-R)v_0'(1) + Rf_1(0,1) + f_{1t}(0,1) - 4f_1(0,1),$$
$$2d(0) = -f_{2x}(0,1) + (G-R)v_0'(0) - Rf_1(0,0) - f_{1t}(0,0) - 4f_1(0,0),$$

then $\hat{f} = (\hat{f}_1, \hat{f}_2)$ *satisfies* ($\mathrm{H_f''}$) *with zero as the initial value.*

THEOREM 5.1. *Assume all of the conditions of Lemma* 5.3. *Then the problem* (*S*), (*BC*), (*IC*) *has a unique classical solution* $(u, v) \in C^1(\overline{D_T})^2$.

Proof. By Theorem 4.3 the problem (5.1)–(5.3) has a unique classical solution $(\hat{u}, \hat{v}) \in C^1(0, T; X)^2$. Moreover, $t \mapsto A(\hat{u}, \hat{v})$ is continuous on $[0, T]$. This $(\hat{u}, \hat{v}) \in C^1(\overline{D_T})^2$. Since a, b, c, d are continuously differentiable, u and v are C^1-functions as well. ∎

THEOREM 5.2. *Assume all of the conditions of Lemma* 5.4. *Then the problem* (*S*), (*BC*), (*IC*) *has a unique classical solution* $(u, v) \in C^2(\overline{D_T})^2$.

Proof. By Theorem 4.4 the problem (5.1)–(5.3) has a unique classical solution $(\hat{u}, \hat{v}) \in C^2(0, T; X)^2$, and $t \mapsto A(\hat{u}, \hat{v})$ is a C^1-function. Hence $\hat{u}_{tx}$ and $\hat{v}_{tx}$ exist, and they are continuous on $\overline{D_T}$. We do not know whether $R\hat{f}(\hat{u}, \hat{v}) \subset D(A)$. However, by (4.37) $\hat{u}_{tx}$ and $\hat{v}_{tx}$ exist on $\overline{D_T}$. Hence $\hat{v}_{tx} = \hat{v}_{xt}$ and $\hat{u}_{tx} = \hat{u}_{xt}$, and we obtain from (S) that $\hat{u}_{xx}, \hat{v}_{xx} \in C(\overline{D_T})$. Thus $(\hat{u}, \hat{v}) \in C^2(\overline{D_T})^2$. Since a, b, c, d are twice continuously differentiable, u and v are C^2-functions as well. ∎

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