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The Coupling of Hyperbolic and Elliptic Boundary Value Problems with Variable Coefficients†

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We consider a one-dimensional coupled problem for elliptic second-order ODEs with *natural* transmission conditions. In one subinterval, the coefficient $\varepsilon > 0$ of the second derivative tends to zero. Then the equation becomes there hyperbolic and the natural transmission conditions are not fulfilled anymore. The solution of the degenerate coupled problem with a flux transmission condition is corrected by an internal boundary layer term taking into account the viscosity ε. By using singular perturbation techniques, we show that the remainders in our first-order asymptotic expansion converge to zero uniformly. Our analysis provides an *a posteriori* correction procedure for the numerical treatment of exterior viscous compressible flow problems with coupled Navier–Stokes/Euler models. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: coupled boundary value problems; hyperbolic–elliptic interaction; singular perturbation and asymptotic expansions

0. Introduction

We consider one-dimensional coupled transmission-boundary value problems of some elliptic ordinary differential equations of second order. In a subdomain, the coefficient of the second-order derivative tends to zero, while the corresponding coefficient in the complementary region is fixed.

Such a coupling of equations is motivated by various applications in Computational Fluid Dynamics, as, for instance, boundary value problems for exterior viscous flows around bodies, modelled by the compressible Navier–Stokes equations. While the viscous terms are of comparable order of magnitude as the convective terms only in a thin region near to the body, further away, the viscous shear stresses are extremely small and become strongly dominated by the convective part. This fact is evident from physics; for a heuristic justification see [3]. Consequently, in the numerical treatment, the viscous terms are often neglected in some distance to the profile. Accordingly, one may consider two different model zones defining a decomposition of the original flow field into a bounded computational domain, where the compressible, stationary

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Navier–Stokes equations are used close to the body, and a corresponding far field with an inviscid model of conservation laws, such as the Euler equations or their linearization (see, e.g. [18, 2, 5]).

The transmission conditions across the interface between the different model zones are to be chosen in such a way, that the fundamental physical laws are respected on one hand, and on the other hand, the resulting coupled problem is well-posed and is consistent with the original problem. One could choose the continuity of the flow variables, but according to the theory of hyperbolic equations, this can be required only at the corresponding inflow interface. However, in accordance with the conservation laws (mass, momentum and energy) the continuity of the normal flux yields a transmission condition on the complete interface: the normal inviscid flux, corresponding to the solution of the hyperbolic equations is set equal to the total flux associated with the solution of the elliptic model, which contains the inviscid as well as the viscous contributions (see, e.g. [14, 18]). But this transmission condition implies that solutions of the coupled hyperbolic–elliptic problem exhibit jumps at the interface (see e.g. [2]), depending on the magnitude of the viscosity terms which are neglected in the far field. Since the solution of the original problem should satisfy the *natural* transmission conditions across a fictitious boundary, we can affirm that the approximate solution to the heterogeneous coupled Navier–Stokes/Euler problem is a *first approximation* of exterior viscous flows taking into account viscosity as well as far field behaviour. This coupled solution needs to be *corrected* by special terms which account for the loss of continuity and maintain the continuity of the normal flux. This method is very useful for the numerical treatment of exterior *compressible viscous* flows. (For *incompressible viscous* flows, recent results have been obtained in [10]; and also in [11, 12], where the incompressible Navier–Stokes equations are coupled with linear models in the far field.)

The above-mentioned boundary layer correction can be analysed in the framework of singular perturbation theory, which we use here for a one-dimensional model case. This can be understood as a simplified version of a two-dimensional coupling of the Navier–Stokes equations where in the far field the viscous terms are strongly dominated by the convective part. The justification is given in [3]; here we only recall the corresponding estimates of the convective and inviscid components within the boundary layer and in the far field and then use the form of the one-dimensional parameter-dependent problem obtained there; see also [2].

The asymptotic analysis of the transmission problem is based on two different specific physical scales: a 'slow' variable x and a rapid variable τ which characterizes the boundary layer. Asymptotic solutions to singularly perturbed boundary value problems for ordinary differential equations have been developed in many contributions (see, for example, the fundamental paper by Vishik and Ljusternik [21], also the monographs [6, 17]). There, the analysis hinges decisively on the specific type of boundary conditions. For initial and boundary value problems see also the book by de Jager [15], where uniform estimates for the remainders as well as for their derivatives are established. Problems with degeneration of elliptic to hyperbolic equations have been treated by Lions [16], and by Eckhaus and de Jager [7].

We investigate here a singularly perturbed one-dimensional elliptic–elliptic *transmission* problem that degenerates to a problem of hyperbolic–elliptic type. This work continues investigations by Gastaldi and Quarteroni in [13] who consider the degenerate problem as the original transmission problem, whereas the elliptic–elliptic coupled problem is viewed as an elliptic regularization obtained by adding artificial 'viscosity' in the hyperbolic region. They show that the solutions of the corresponding regularized problems converge in L^2 to the solution of the degenerate problem as the artificial viscosity parameter tends to zero. We take here a different point of view: we use appropriate boundary layer functions as a viscosity correction of the solution to the simplified but degenerate model problem. In the case of essentially bounded right-hand sides, which are continuous at the interface, we obtain beyond the L^2-convergence in [13] *uniform rigorous estimates* for the remainders.

Our *a posteriori* viscosity correction is explicitly determined by the solution of the degenerate simplified problem. This allows the approximation of the solution to the original problem including viscosity with any given accuracy, also numerically. Numerical experiments agree very well with our analysis, and are presented in the last part of this paper.

Our one-dimensional analysis allows an extension to higher dimensions since the explicit computation of a boundary layer viscosity correction for the approximate solution of the coupled Navier–Stokes and Euler model provides *a posteriori* an asymptotically improved approximation of the full Navier–Stokes equations.

The paper is organized as follows. In section 1 we recall from [3] some aspects which give a justification for the use of the heterogeneous coupling in CFD and motivate the analysis of a simplified one-dimensional problem. After the problem setting in section 2, section 3 is devoted to formal asymptotic expansions of the coupled problem. In section 4 we present uniform estimates of the remainder functions. In section 5 we present some results of the numerical computations.

1. Motivation of the paper

1.1. *Justification of the coupling of Navier–Stokes/Euler models*

The motion of fluids is modelled by nonlinear partial differential equations derived from the conservation of mass, momentum and energy. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain representing a given simply connected wing section. The conservation laws lead in the exterior to the complete system of viscous compressible flow consisting of the continuity equation, the *Navier–Stokes equations* and the energy equation. In the stationary case, these can be written in the dimensionless form

$$\sum_{i=1}^{2} \frac{\partial \mathbf{f}_i(\mathbf{w})}{\partial x_i} = \frac{1}{Re} \sum_{i=1}^{2} \frac{\partial \mathbf{R}_i(\mathbf{w}, \nabla_{\mathbf{x}} \mathbf{w})}{\partial x_i} \quad \text{in } \Omega^c = \mathbb{R}^2 \backslash \Omega, \tag{1.1}$$

where $\mathbf{w} = (\rho, \ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ collects the dimensionless conservative variables, with ρ the density, $\mathbf{v} = (v_1, v_2)^{\mathrm{T}}$ the velocity vector and e the total energy. The inviscid (Euler) fluxes $\mathbf{f}_i$ and the viscous terms $\mathbf{R}_i$ describe the convective and the viscous parts, respectively, and are given by

$$\mathbf{f}_i(\mathbf{w}) = (\rho v_i, \rho v_i v_1 + \delta_{i1} p, \rho v_i v_2 + \delta_{i2} p, (e + p) v_i)^{\mathrm{T}}, \tag{1.2}$$

$$\mathbf{R}_i(\mathbf{w}, \nabla_{\mathbf{x}} \mathbf{w}) = (0, \tau_{i1}, \tau_{i2}, \tau_{i1} v_1 + \tau_{i2} v_2 + k \partial\theta/\partial x_i)^{\mathrm{T}}, \quad i = 1, 2. \tag{1.3}$$

The functions p, θ and τ_{ij} denote the dimensionless pressure, absolute temperature and the dimensionless components of the viscous part of the stress tensor, respectively. The latter is defined by $\tau_{ij} = \lambda \operatorname{div} \mathbf{v}\, \delta_{ij} + \mu(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ for $i, j = 1, 2$. Here λ, μ and k are the dimensionless viscosity coefficients and the heat conductivity, respectively. $Re := U^* L^* \rho^*/\mu^*$ is the characteristic *Reynolds number*, where U^*, L^*, ρ^* and μ^* are reference quantities related to the free-stream flow and to the geometry of the profile. The system is completed by the equations of state expressing constitutive thermodynamic relations.

Since a real fluid adheres to the body, large shear stresses arise inside Prandtl's boundary layer near to the profile. Even for high Reynolds numbers, within the boundary layer, the viscous terms $(1/Re)\sum_{i=1}^{2} \partial \mathbf{R}_i(\mathbf{w}, \nabla_{\mathbf{x}} \mathbf{w})/\partial x_i$ are signifiant, whereas away from the aerofoil they decrease drastically. In [3] we calculate the order of magnitude of the different terms in the dimensionless momentum equations with respect to the thickness δ of Prandtl's boundary layer. The terms under consideration are the convective parts in the Navier–Stokes equations,

$$\begin{aligned}
\sum_{i=1}^{2} \frac{\partial \mathbf{f}_{i,2}(\mathbf{w})}{\partial x_i} &= \rho\left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2}\right) + \frac{\partial p}{\partial x_1},\\
\sum_{i=1}^{2} \frac{\partial \mathbf{f}_{i,3}(\mathbf{w})}{\partial x_i} &= \rho\left(v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2}\right) + \frac{\partial p}{\partial x_2},
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
\frac{1}{Re} \sum_{i=1}^{2} \frac{\partial \mathbf{R}_{i,2}(\mathbf{w}, \nabla_{\mathbf{x}} \mathbf{w})}{\partial x_i} = \frac{1}{Re} \Bigg\{ & \frac{\partial}{\partial x_1}\left[\lambda\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) + 2\mu \frac{\partial v_1}{\partial x_1}\right] \\
& + \frac{\partial}{\partial x_2}\left[\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right]\Bigg\},
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
\frac{1}{Re} \sum_{i=1}^{2} \frac{\partial \mathbf{R}_{i,3}(\mathbf{w}, \nabla_{\mathbf{x}} \mathbf{w})}{\partial x_i} = \frac{1}{Re} \Bigg\{ & \frac{\partial}{\partial x_1}\left[\mu\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\right] \\
& + \frac{\partial}{\partial x_2}\left[\lambda\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) + 2\mu \frac{\partial v_2}{\partial x_2}\right]\Bigg\}.
\end{aligned} \tag{1.6}$$

We summarize the estimates from [3] in Table 1.

Since within Prandtl's boundary layer the dissipative parts in both equations are of the same order as the convective parts, there *the viscous terms must be considered for the analytical and numerical treatment of the problem.* In contrast to the boundary layer, in the far field both convective parts in the Navier–Stokes equations are of order one, whereas the viscous terms are of the order $\mathcal{O}(\delta^2) = \mathcal{O}(1/Re)$. This allows us to use the Euler model as a first approximation in the far field, where the viscous terms are

Table 1. Orders of magnitude of different parts of the momentum equations

First momentum equation	Prandtl's boundary layer	Convective part $(1.4)_1$	$\mathcal{O}(1)$
		Dissipative part (1.5)	$\mathcal{O}(1)$
	Far field	Convective part $(1.4)_1$	$\mathcal{O}(1)$
		Dissipative part (1.5)	$\mathcal{O}(\delta^2)$
Second momentum equation	Prandtl's boundary layer	Convective part $(1.4)_2$	$\mathcal{O}(\delta)$
		Dissipative part (1.6)	$\mathcal{O}(\delta)$
	Far field	Convective part $(1.4)_2$	$\mathcal{O}(1)$
		Dissipative part (1.6)	$\mathcal{O}(\delta^2)$

neglected since *Re* is large. This motivates us to analyse the coupling of two elliptic boundary value problems where the viscosity coefficient is small in one subregion. The corresponding degenerate (reduced) hyperbolic–elliptic transmission problem corresponds to the Navier–Stokes/Euler coupling in CFD.

In addition to the governing equations, the conservation laws imply the transmission conditions

$$\sum_{i=1}^{2} \mathbf{f}_i(\mathbf{w}_{\mathrm{E}})n_i = -\frac{1}{Re}\sum_{i=1}^{2} \mathbf{R}_i(\mathbf{w}_{\mathrm{NS}}, \nabla_{\mathbf{x}}\mathbf{w}_{\mathrm{NS}})n_i + \sum_{i=1}^{2} \mathbf{f}_i(\mathbf{w}_{\mathrm{NS}})n_i, \tag{1.7}$$

where $\mathbf{n} = (n_1, n_2)$ denotes the unit normal vector of the interface. These conditions are obtained from the weak formulation of the conservation laws, where in the Euler region the viscous fluxes are neglected.

1.2. *Dimensional reduction of the Navier–Stokes equations and application to flows around the aerofoil*

In [2, 3] we also give a motivation for the asymptotic analysis of a one-dimensional coupled problem, showing that the full two-dimensional Navier–Stokes problem can *locally* be reduced to a one-dimensional parameter-dependent problem. Using the rotational invariance of the Euler system (see e.g. [8]), we write the Navier–Stokes equations in the form

$$\frac{\partial \mathbf{f}_1(\mathbf{q})}{\partial \tilde{x}_1} = \frac{\partial \mathbf{R}_1(\mathbf{q}, \partial\mathbf{q}/\partial\tilde{x}_1, \partial\mathbf{q}/\partial\tilde{x}_2)}{\partial \tilde{x}_1}, \tag{1.8}$$

where $(\tilde{x}_1, \tilde{x}_2)^{\mathrm{T}}$ are new Cartesian co-ordinates given by

$$\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix} = \begin{pmatrix} n_1 & n_2\\ -n_2 & n_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \boldsymbol{\sigma} \quad \text{with}$$

$$\mathbf{n} = (n_1, n_2)^{\mathrm{T}} \in \mathbb{R}^2, \quad |\mathbf{n}| = 1 \quad \text{and } \boldsymbol{\sigma} \in \mathbb{R}^2, \tag{1.9}$$

and where the new state vector $\mathbf{q}$ is defined by

$$\mathbf{q} = (\rho, \rho(n_1 v_1 + n_2 v_2), \rho(-n_2 v_1 + n_1 v_2), e)^{\mathrm{T}}.$$

The convective and dissipative fluxes $\mathbf{f}_1$ and $\mathbf{R}_1$ are differentiated in (1.8) only with respect to $\tilde{x}_1$. However, the partial derivative $\partial \mathbf{q}/\partial \tilde{x}_2$ still appears as one of the arguments in the viscous flux function $\mathbf{R}_1$. This difficulty can be handled by considering this quantity as an additional parameter in the local asymptotic analysis of the problem.

In [2, 3], we further reduce (1.8) to a new system of the form

$$-\frac{\mathrm{d}\mathbf{\Psi}_1}{\mathrm{d}s}(s) + \frac{\mathrm{d}\mathbf{\Phi}_1}{\mathrm{d}s}(s) = 0 \quad \text{for} \quad s \in I := [a, c] \subset \mathbb{R}, \tag{1.10}$$

with appropriate vector-valued functions $\mathbf{\Psi}_1, \mathbf{\Phi}_1 : [a, c] \to \mathbb{R}^4$, modelling the viscous and the inviscid fluxes, respectively. Here, I is a parametric interval of a curve $\gamma = \{(\gamma_1(s), \gamma_2(s)) | s \in I\} \subset \mathbb{R}^2$, starting at some given point $P = \gamma(a)$ situated in the far field, intersecting the aerofoil at some point $R = \gamma(c) \in \partial\Omega$, and intersecting the boundary layer at some point $Q = \gamma(b)$ that defines $b \in (a, c)$.

The dimensionally reduced Navier–Stokes equations on the interval $[a, b]$ now correspond to a system of elliptic equations where the viscosity terms are dominated by the convective terms, whereas on the interval $[b, c]$ (note that the curved segment $[QR]$ is completely contained in the boundary layer) they correspond to an elliptic system where the viscous terms have the same order of magnitude as the convective ones. The viscosity coefficient appears as a small parameter ε in the equation on $[a, b]$, while the degenerate transmission problem is hyperbolic on $[a, b]$ and elliptic on $[b, c]$.

In what follows we shall analyse a simplified model of this transmission problem in the form of a scalar equation which corresponds to a decoupled system. The decoupling is always possible for the hyperbolic convective part, whereas the dissipative terms cannot be diagonalized, in general, except in case of vanishing shear stresses. In this particular situation, our simplified model will provide us with correction terms for each component of the degenerate model.

2. Model boundary value problem

We consider the following:

Coupled elliptic–elliptic transmission-boundary value problem ($\mathbf{P}_\varepsilon$). *Let*

(i) a, b, c *be real numbers, with* $a < b < c$;
(ii) α, β *be real-valued functions, defined on* $[a, c]$ *with* $\alpha \in H^1(a, c)$, $\alpha(x) \geqslant \alpha_0 > 0$ *and with* $\beta \in L^\infty(a, c)$ *and* $\beta(x) \geqslant 0$;
(iii) f, g *be real-valued functions,* $f \in L^2(a, b)$, $g \in L^2(b, c)$;
(iv) $\varepsilon > 0$ *be a small parameter and* $\mu > 0$ *be a given constant representing the viscosity.*

Find real-valued functions $u_\varepsilon \in H^2(a, b)$, $v_\varepsilon \in H^2(b, c)$ *such that the equations*

$$-\varepsilon u_\varepsilon''(x) + \alpha(x) u_\varepsilon'(x) + \beta(x) u_\varepsilon(x) = f(x) \quad \text{for} \quad x \in (a, b), \tag{2.1}$$

$$-\mu v_\varepsilon''(x) + \alpha(x) v_\varepsilon'(x) + \beta(x) v_\varepsilon(x) = g(x) \quad \text{for} \quad x \in (b, c), \tag{2.2}$$

and the boundary and transmission conditions

$$u_\varepsilon(a) = v_\varepsilon(c) = 0 \quad \textit{and} \quad u_\varepsilon(b) = v_\varepsilon(b), \quad \varepsilon u'_\varepsilon(b) = \mu v'_\varepsilon(b) \tag{2.3}$$

are satisfied.

As usual, we impose the continuity of the solution and of the normal flux as *natural* transmission conditions for this elliptic coupled problem. For $\varepsilon = 0$, however, we employ the

Reduced (degenerate) hyperbolic–elliptic problem ($\mathbf{P_0}$). *Find real-valued functions $u \in H^1(a, b)$ and $v \in H^2(b, c)$ such that the equations*

$$\alpha(x)u'(x) + \beta(x)u(x) = f(x) \quad \textit{for} \quad x \in (a, b), \tag{2.4}$$

$$-\mu v''(x) + \alpha(x)v'(x) + \beta(x)v(x) = g(x) \quad \textit{for} \quad x \in (b, c), \tag{2.5}$$

$$u(a) = 0, \quad v(c) = 0, \quad \alpha(b)u(b) = -\mu v'(b) + \alpha(b)v(b) \tag{2.6}$$

are satisfied in the distributional sense.

Note that the correct transmission condition at b will be the conservation of mass, which does *not* coincide with the usual transmission condition of continuous velocity chosen in traditional singular perturbation theory. Our condition corresponds to the transmission condition (1.7) of the heterogeneous coupling in CFD, and allows the solution of the coupled hyperbolic–elliptic problem to be discontinuous at the common interface. The interface condition for the one-dimensional model has also been used in [13], obtained by the weak formulation of the coupled problem.

Here we will consider a first-order approximation of the solution $(u_\varepsilon, v_\varepsilon)$ to the elliptic–elliptic coupled boundary value problem about the solution (u, v) of the degenerate problem, and we will show that the use of an appropriate boundary layer function enables us to prove—beyond the convergence result from [13], i.e.

$$(u_\varepsilon - u, v_\varepsilon - v) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^2(a, b) \times L^2(b, c),$$

—rigorous uniform estimates for the remainders in the asymptotic expansion.

Let us first briefly discuss the solvability of the boundary value problems introduced above. In accordance with the practical background, we will assume that, in addition to

$$\alpha(x) \geqslant \alpha_0 > 0 \quad \text{for all} \quad x \in [a, c],$$

$$\beta(x) \geqslant 0 \quad \text{for almost every } x \in [a, c] \tag{2.7}$$

the functions α and β satisfy also the coercivity condition

$$\beta(x) - \tfrac{1}{2}\alpha'(x) \geqslant 0 \quad \text{for almost every} \quad x \in [a, c]. \tag{2.8}$$

Solvability of the elliptic–elliptic coupled boundary value problem (P_ε).

The weak form of this problem is to find $w_\varepsilon \in H_0^1(a, c)$ such that

$$a_\varepsilon(w_\varepsilon, \varphi) := \int_a^c \mu_\varepsilon w_\varepsilon' \varphi' \, dx + \int_a^c (\alpha w_\varepsilon' + \beta w_\varepsilon) \varphi \, dx$$

$$= \int_a^b f\varphi \, dx + \int_b^c g\varphi \, dx \quad \text{for all } \varphi \in H_0^1(a, c).$$

Here, $w_\varepsilon|_{[a,b]} = u_\varepsilon$, $w_\varepsilon|_{[b,c]} = v_\varepsilon$, $\mu_\varepsilon|_{[a,b]} = \varepsilon$ and $\mu_\varepsilon|_{[b,c]} = \mu$. The bilinear form a_ε is continuous on $H_0^1(a, c)$ and satisfies for all $w \in H_0^1(a, c)$ the coerciveness inequality

$$a_\varepsilon(w, w) \geqslant \gamma_0(\varepsilon) \| w \|_{H^1(a,c)}^2 \quad \text{with} \quad \gamma_0(\varepsilon) := \min \left\{ \frac{\varepsilon}{2}, \frac{\varepsilon}{2(c-a)^2} \right\}. \tag{2.9}$$

Clearly, the coerciveness deteriorates as $\varepsilon \to 0$. The well-known Lax–Milgram theorem implies the existence of a unique solution $w_\varepsilon \in H_0^1(a, c)$, hence $u_\varepsilon \in H^1(a, b)$ and $v_\varepsilon \in H^1(b, c)$. Using equations (2.1), (2.2), we also get immediately

$$(u_\varepsilon, v_\varepsilon) \in H^2(a, b) \times H^2(b, c). \tag{2.10}$$

Solvability of the degenerate boundary value problem (P_0).

The problem (2.4) with $u(a) = 0$ admits a unique solution $u \in H^1(a, b)$, hence $u \in C[a, b]$. With $\gamma := \alpha(b) u(b)$, the function v has to solve the boundary value problem consisting of (2.5) with $v(c) = 0$ and the boundary condition

$$-\mu v'(b) + \alpha(b) v(b) = \gamma. \tag{2.11}$$

In a first step we homogenize the boundary condition (2.11). If v is a solution of the above problem, then the function $\tilde{v}$: $[b, c] \to \mathbb{R}$ defined by

$$\tilde{v}(x) := v(x) + \frac{\gamma}{\alpha(b)(c-b) + \mu}(x - c) \quad \text{for} \quad x \in [b, c]$$

satisfies the equation

$$-\mu \tilde{v}''(x) + \alpha(x) \tilde{v}'(x) + \beta(x) \tilde{v}(x) = \tilde{g}(x) \tag{2.12}$$

with $\tilde{g} \in L^2(b, c)$ given by $\tilde{g}(x) := g(x) + [\gamma\alpha(x) + \gamma(x - c)\beta(x)]/[\alpha(b)(c - b) + \mu]$ for $x \in [b, c]$, together with the homogeneous boundary conditions

$$\tilde{v}(c) = 0 \quad \text{and} \quad -\mu \tilde{v}'(b) + \alpha(b) \tilde{v}(b) = 0. \tag{2.13}$$

We further define the space $(V, \| \cdot \|_V)$ by

$$V := \{ \tilde{v} \in H^1(b, c) : \tilde{v}(c) = 0 \} \quad \text{with} \quad \| \tilde{v} \|_V := \left[\| \tilde{v} \|_{H^1(b,c)}^2 + \frac{1}{2} \alpha(b) \tilde{v}^2(b) \right]^{1/2}. \tag{2.14}$$

This is a Hilbert space, because the V-norm is equivalent to the $\|\cdot\|_{H^1(a,b)}$-norm: one has $\|\tilde{v}\|_{H^1(a,b)} \leqslant \|\tilde{v}\|_V \leqslant [1+\alpha(b)(c-b)/2]^{1/2}\,\|\tilde{v}\|_{H^1(a,b)}$ for all $\tilde{v}\in V$. The variational form of the problem (2.12), (2.13) is to find $\tilde{v}\in V$, such that

$$a(\tilde{v},\psi) := \mu\int_b^c \tilde{v}'\psi'\,\mathrm{d}x + \int_b^c (\alpha\tilde{v}' + \beta\tilde{v})\,\psi\,\mathrm{d}x + \alpha(b)\,\tilde{v}(b)\,\psi(b)$$

$$= \int_b^c \tilde{g}\psi\,\mathrm{d}x \quad \text{for all} \quad \psi\in V.$$

The bilinear form a is continuous on V and has the form

$$a(\tilde{v},\tilde{v}) = \mu\int_b^c (\tilde{v}'(x))^2\,\mathrm{d}x + \int_b^c \left[\beta - \frac{\alpha'}{2}\right](x)\,\tilde{v}^2(x)\,\mathrm{d}x + \frac{1}{2}\,\alpha(b)\,\tilde{v}^2(b).$$

Hence, with the Poincaré inequality as well as the condition (2.8), we get

$$a(\tilde{v},\tilde{v}) \geqslant \min\left\{\frac{\mu}{2}, \frac{\mu}{2(c-b)^2}\right\}\|\tilde{v}\|^2_{H^1(b,c)}$$

$$+\frac{1}{2}\,\alpha(b)\,\tilde{v}^2(b) \geqslant \gamma_0\,\|\tilde{v}\|^2_V \quad \text{for all} \quad \tilde{v}\in V, \tag{2.15}$$

with $\gamma_0 := \min\{1, \mu/2, \mu/2(c-b)^2\} > 0$. The Lax–Milgram theorem implies existence and uniqueness of a solution $\tilde{v}\in V$ and, consequently, the problem (2.5), (2.11) with $v(c)=0$ admits a unique solution $v\in H^1(b,c)$. We use once more equation (2.5) and find $v''\in L^2(b,c)$, hence

$$(u,v)\in H^1(a,b)\times H^2(b,c). \tag{2.16}$$

3. Formal asymptotic expansion

We now represent the solution $(u_\varepsilon, v_\varepsilon)$ of the problem (P_ε) as a perturbation of the solution (u_0, v_0) of the problem (P_0) in the form

$$u_\varepsilon(x) = u_0(x) + \ell_\varepsilon(x) + r_\varepsilon(x) \quad \text{for} \quad x\in[a,b], \tag{3.1}$$

$$v_\varepsilon(x) = v_0(x) + s_\varepsilon(x) \quad \text{for} \quad x\in[b,c]. \tag{3.2}$$

Here, ℓ_ε denotes an appropriate boundary layer function, which will be determined in what follows, whereas r_ε and s_ε are the remainders in the asymptotic expansion. The function v_ε has no corrector since the equation (2.2) does not change its type by regularization. This agrees with the corresponding Vishik–Ljusternik analysis in [21].

3.1. *Case of vanishing damping* $\beta\equiv 0$, $\alpha'\leqslant 0$

Here, we apply the analysis by Vishik and Ljusternik. We introduce the rapid variable

$$\tau := (b-x)/\varepsilon \tag{3.3}$$

and define the boundary layer function $\tilde{\ell}_\varepsilon(\tau)$ by $\tilde{\ell}_\varepsilon(\tau) := \ell_\varepsilon(b - \varepsilon\tau)$ for $0 \leqslant \tau \leqslant (b-a)/\varepsilon$. The use of two scales of variables allows us to obtain a better description of the solution in the boundary layer region. Inserting $(u_\varepsilon, v_\varepsilon)$ from (3.1), (3.2) into (2.1), (2.2) and using the relations $\ell'_\varepsilon(x) = -(1/\varepsilon)\dot{\tilde{\ell}}_\varepsilon$, $\ell''_\varepsilon(x) = (1/\varepsilon^2)\ddot{\tilde{\ell}}_\varepsilon$, we find the equations

$$-\varepsilon u_0''(x) - \frac{1}{\varepsilon}\ddot{\tilde{\ell}}_\varepsilon(\tau) - \varepsilon r_\varepsilon''(x) + \alpha(x)u_0'(x) - \frac{1}{\varepsilon}\alpha(b-\varepsilon\tau)\dot{\tilde{\ell}}_\varepsilon(\tau) + \alpha(x)r_\varepsilon'(x) = f(x),$$

$$x = b - \varepsilon\tau \in (a, b), \tag{3.4}$$

$$-\mu v_0''(x) - \mu s_\varepsilon''(x) + \alpha(x)v_0'(x) + \alpha(x)s_\varepsilon'(x) = g(x), \quad x \in (b, c). \tag{3.5}$$

In a first step we determine the boundary layer function ℓ_ε. We collect the terms which depend on the rapid variable τ and are of order ε^{-1} in (3.4). We set

$$\ddot{\tilde{\ell}}_\varepsilon(\tau) + \alpha(b-\varepsilon\tau)\dot{\tilde{\ell}}_\varepsilon(\tau) = 0, \tag{3.6}$$

and, in consequence,

$$\ell_\varepsilon(x) = C + C_1\int_0^{(b-x)/\varepsilon} \mathrm{e}^{-\int_0^s \alpha(b-\varepsilon u)\,\mathrm{d}u}\,\mathrm{d}s \quad \text{for} \quad x \in [a, b], \quad \text{with} \quad C, C_1 \in \mathbb{R}.$$

Choosing the constants C and C_1 in such a manner, that $\ell_\varepsilon \not\equiv 0$ satisfies $\lim_{\varepsilon\to 0}\ell_\varepsilon(x) = 0$ for every $x \in [a, b)$, we obtain the relation $C = -C_1/\alpha(b)$. The constant C_1 will be determined by appropriate matching. Since (u_0, v_0) is chosen as the solution of the degenerate problem satisfying the transmission condition

$$\alpha(b)u_0(b) = -\eta v_0'(b) + \alpha(b)u_0(b), \tag{3.7}$$

u_0 and v_0 do not join continuously at $x = b$, in general. The jump $v_0(b) - u_0(b)$ will be transfered to the boundary layer function ℓ_ε in order to match continuity in (2.3). This yields the complete form of the boundary layer function:

$$\ell_\varepsilon(x) = (v_0(b) - u_0(b))\left[1 - \alpha(b)\int_0^{(b-x)/\varepsilon} \mathrm{e}^{-\int_0^s \alpha(b-\varepsilon u)\,\mathrm{d}u}\,\mathrm{d}s\right] \quad \text{for} \quad x \in [a, b]. \tag{3.8}$$

We note that then the remainders r_ε and s_ε join continuously at $x = b$. The second transmission condition in (2.3), together with the representations (3.1), (3.2) and with ℓ_ε defined in (3.8), leads to the relation

$$\varepsilon u_0'(b) + \alpha(b)[v_0(b) - u_0(b)] + \varepsilon r_\varepsilon'(b) = \mu v_0'(b) + \mu s_\varepsilon'(b)$$

for the flux. Here we use (3.7) and then "match" the terms of order ε to find the flux condition for the remainders:

$$\varepsilon r_\varepsilon'(b) + \varepsilon u_0'(b) = \mu s_\varepsilon'(b). \tag{3.9}$$

Finally, we establish the boundary conditions for the remainders. The solution u_0 of the degenerate coupled problem satisfies the condition $u_0(a) = 0$. To impose the boundary condition $u_\varepsilon(a) = 0$, we require

$$\Omega(\varepsilon) := r_\varepsilon(a) = -\ell_\varepsilon(a)$$

$$= -(v_0(b) - u_0(b))\left[1 - \alpha(b)\cdot\int_0^{(b-a)/\varepsilon} \mathrm{e}^{-\int_0^s \alpha(b-\varepsilon u)\,\mathrm{d}u}\,\mathrm{d}s\right]. \tag{3.10}$$

Note that $v_0(b) - u_0(b)$ and $\Omega(\varepsilon)$ depend on the right-hand sides. An evaluation for $r_\varepsilon(a)$ can be obtained as follows.

Lemma 3.1. *There exists a constant $M_1 > 0$ such that*

$$|\ell_\varepsilon(a)| \leqslant M_1 \varepsilon \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. Will be given in Appendix A.

At $x = c$ we impose $v_0(c) = 0$. From $v_\varepsilon(c) = 0$ and (3.2) we find $s_\varepsilon(c) = 0$. Collecting all these conditions, we now obtain the

Transmission-boundary value problem for the remainders. *Find real-valued functions $(r_\varepsilon, s_\varepsilon)$ such that the equations and the boundary and transmission conditions*

$$-\varepsilon r_\varepsilon''(x) + \alpha(x) r_\varepsilon'(x) = \varepsilon u_0''(x) \quad \textit{for} \quad x \in (a, b), \qquad r_\varepsilon(a) = \Omega(\varepsilon), \tag{3.11}$$

$$(\mathrm{R}_\varepsilon): \quad -\mu s_\varepsilon''(x) + \alpha(x) s_\varepsilon'(x) = 0 \quad \textit{for} \quad x \in (b, c), \qquad s_\varepsilon(c) = 0, \tag{3.12}$$

$$r_\varepsilon(b) = s_\varepsilon(b), \quad \varepsilon r_\varepsilon'(b) + \varepsilon u_0'(b) = \mu s_\varepsilon'(b) \tag{3.13}$$

are satisfied.

For estimating $(r_\varepsilon, s_\varepsilon)$ for $\varepsilon \to 0$, we first decompose the solution as follows:

$$r_\varepsilon(x) = r_{1\varepsilon}(x) + r_{2\varepsilon}(x) \quad \text{for} \quad x \in [a, b],$$

$$s_\varepsilon(x) = s_{1\varepsilon}(x) + s_{2\varepsilon}(x) \quad \text{for} \quad x \in [b, c], \tag{3.14}$$

where $(r_{1\varepsilon}, s_{1\varepsilon})$ and $(r_{2\varepsilon}, s_{2\varepsilon})$ are the corresponding solutions of the auxiliary boundary value problems (here the index ε is omitted):

$$-\varepsilon r_1''(x) + \alpha(x) r_1'(x) = 0 \quad \text{for} \quad x \in (a, b), \quad r_1(a) = \Omega(\varepsilon), \tag{3.15}$$

$$(\mathrm{R}_1): \quad -\mu s_1''(x) + \alpha(x) s_1'(x) = 0 \quad \text{for} \quad x \in (b, c), \quad s_1(c) = 0, \tag{3.16}$$

$$r_1(b) = s_1(b), \quad \varepsilon r_1'(b) + \varepsilon u_0'(b) = \mu s_1'(b); \tag{3.17}$$

and

$$-\varepsilon r_2''(x) + \alpha(x) r_2'(x) = \varepsilon u_0''(x) \quad \text{for} \quad x \in (a, b), \quad r_2(a) = 0, \tag{3.18}$$

$$(\mathrm{R}_2): \quad -\mu s_2''(x) + \alpha(x) s_2'(x) = 0 \quad \text{for} \quad x \in (b, c), \quad s_2(c) = 0, \tag{3.19}$$

$$r_2(b) = s_2(b), \quad \varepsilon r_2'(b) = \mu s_2'(b). \tag{3.20}$$

3.2. *Case of non-vanishing damping* $\beta \geqslant 0, \beta - \frac{1}{2}\alpha' \geqslant 0$

In the usual approach, the boundary layer function ℓ_ε would be set identically zero. Here, however, we still need the boundary layer function ℓ_ε from the case $\beta \equiv 0$ also for the case $\beta \geqslant 0$ since it still takes care of the jump between u_0 and v_0 at $x = b$. The presence of this corrector does not contradict the L^2-convergence (which was proved in [13]), because the L^2-norm of ℓ_ε tends to zero for $\varepsilon \to 0$. The following lemma is proved in Appendix B.

Lemma 3.2. *There exists a constant* $M_2 > 0$, *such that*

$$\|\ell_\varepsilon\|_{L^2(a,b)} \leqslant M_2\sqrt{\varepsilon} \quad \textit{for every} \quad \varepsilon > 0. \tag{3.21}$$

The equation for r_ε is now to be modified by the term $\beta(x)\ell_\varepsilon(x)$:

$$-\varepsilon r_\varepsilon''(x) + \alpha(x)r_\varepsilon'(x) + \beta(x)r_\varepsilon(x) = -\beta(x)\ell_\varepsilon(x) + \varepsilon u_0''(x) \quad \text{for} \quad x \in (a, b), \tag{3.22}$$

whereas the equation for s_ε becomes

$$-\mu s_\varepsilon''(x) + \alpha(x)s_\varepsilon'(x) + \beta(x)s_\varepsilon(x) = 0 \quad \text{for} \quad x \in (b, c). \tag{3.23}$$

The boundary and transmission conditions in (3.11)–(3.13) remain the same. The remainder functions are decomposed as follows:

$$\begin{aligned} r_\varepsilon(x) &= r_{3\varepsilon}(x) + r_{4\varepsilon}(x) \quad \text{for} \quad x \in [a, b], \\ s_\varepsilon(x) &= s_{3\varepsilon}(x) + s_{4\varepsilon}(x) \quad \text{for} \quad x \in [b, c], \end{aligned} \tag{3.24}$$

where the functions $(r_{3\varepsilon}, s_{3\varepsilon})$ and $(r_{4\varepsilon}, s_{4\varepsilon})$ are the respective solutions of the auxiliary boundary value problems (again, the index ε is omitted) :

$$(\mathrm{R}_3): \quad \begin{aligned} &-\varepsilon r_3''(x) + \alpha(x)r_3'(x) = \varepsilon u_0''(x) \quad \text{for} \quad x \in (a, b), \qquad r_3(a) = \Omega(\varepsilon), \\ &-\mu s_3''(x) + \alpha(x)s_3'(x) = 0, \quad \text{for} \quad x \in (b, c), \quad s_3(c) = 0, \\ &r_3(b) = s_3(b), \quad \varepsilon r_3'(b) + \varepsilon u_0'(b) = \mu s_3'(b); \end{aligned} \tag{3.25}$$

and

$$(\mathrm{R}_4): \quad \begin{aligned} &-\varepsilon r_4''(x) + \alpha(x)r_4'(x) + \beta(x)r_4(x) = -\beta(x)(r_3(x) + \ell_\varepsilon(x)) \\ &\quad \text{for} \quad x \in (a, b), \quad r_4(a) = 0, && (3.26) \\ &-\mu s_4''(x) + \alpha(x)s_4'(x) + \beta(x)s_4(x) = -\beta(x)s_3(x) \\ &\quad \text{for} \quad x \in (b, c), \quad s_4(c) = 0, && (3.27) \\ &r_4(b) = s_4(b), \qquad \varepsilon r_4'(b) = \mu s_4'(b). && (3.28) \end{aligned}$$

In the next subsection, we collect existence and regularity results for the boundary value problems of the remainders which allow us to obtain the desired rigorous maximum norm estimates for r_ε and s_ε in section 4.

3.3 *Existence and regularity*

The problem for (u_0, v_0) coincides with (2.4)–(2.6). Assuming (2.8) we have

$$(u_0, v_0) \in H^1(a,b) \times H^2(b,c). \tag{3.29}$$

Let us briefly investigate the case of non-vanishing damping. We deal with the transmission-boundary value problem for $(r_\varepsilon, s_\varepsilon)$, consisting of the equations (3.22), (3.23) together with the transmission and boundary conditions from the problem with vanishing damping (3.11)–(3.13). The latter is a particular case and the analysis can be easily modified.

Note that due to $\beta \in L^\infty(a,b)$ and $\ell_\varepsilon \in L^2(a,b)$ on the right-hand side in (3.22), there follows

$$-\beta\ell_\varepsilon + \varepsilon u_0'' \in L^2(a,b) \Leftrightarrow u_0'' \in L^2(a,b). \tag{3.30}$$

If $f \in H^1(a,b)$, then the solution u_0 of $\alpha(x)\, u_0'(x) + \beta(x) u_0(x) = f(x)$ in $[a,b]$ would belong to $H^2(a,b)$, hence $-\beta\ell_\varepsilon + \varepsilon u_0'' \in L^2(a,b)$ and, consequently, the solution $(r_\varepsilon, s_\varepsilon)$ would exist in the classical sense; and in a similar manner as for the solution $(u_\varepsilon, v_\varepsilon)$ of (P_ε), one would get $(r_\varepsilon, s_\varepsilon) \in H^2(a,b) \times H^2(b,c)$.

However, for $f \in L^2(a,b)$, we have less regularity. We first homogenize the boundary conditions by introducing the new unknown function

$$\tilde{r}_\varepsilon(x) := r_\varepsilon(x) + Ax^2 + Bx + C, \quad x \in [a,b],$$

where A, B, C are uniquely determined by the conditions

$$\tilde{r}_\varepsilon(a) = 0, \qquad \tilde{r}_\varepsilon(b) = s_\varepsilon(b), \qquad \varepsilon\tilde{r}_\varepsilon'(b) + \varepsilon u_0'(b) = \mu s_\varepsilon'(b). \tag{3.31}$$

Having A, B, C available, the function $\tilde{r}_\varepsilon$ satisfies the differential equation

$$-\varepsilon\tilde{r}_\varepsilon''(x) + \alpha(x)\tilde{r}_\varepsilon'(x) + \beta(x)\tilde{r}_\varepsilon(x) = -\beta(x)\ell_\varepsilon(x) + \varepsilon u_0''(x) + h(x), \quad x \in (a,b)$$

with $h(x) := [-2A\varepsilon + \alpha(x)(2Ax + B) + \beta(x)(Ax^2 + Bx + C)] \in L^2(a,b)$, together with the boundary and transmission conditions (3.31). The corresponding variational problem for the "remainders" $(\tilde{r}_\varepsilon, s_\varepsilon)$ reads as follows:

Find $w_\varepsilon = (\tilde{r}_\varepsilon, s_\varepsilon) \in H_0^1(a,c)$ *such that*

$$\tilde{a}(w_\varepsilon, \varphi) = d(\varphi) \quad \textit{for all} \quad \varphi^\varepsilon H_0^1(a,c), \tag{3.32}$$

where the bilinear form $\tilde{a}$ on $H_0^1(a,c)$ is defined by

$$\tilde{a}(w_\varepsilon, \varphi) := \varepsilon \int_a^b \tilde{r}_\varepsilon'(x)\varphi'(x)\,\mathrm{d}x + \mu \int_b^c s_\varepsilon'(x)\varphi'(x)\,\mathrm{d}x$$

$$+ \int_a^c [\alpha(x) w_\varepsilon'(x) + \beta(x) w_\varepsilon(x)]\varphi(x)\,\mathrm{d}x \tag{3.33}$$

and where $d \in H^{-1}(a, c)$ is defined by

$$d(\varphi) := -\int_a^b \beta(x)\ell_\varepsilon(x)\varphi(x)\,dx - \varepsilon\int_a^b \frac{f(x) - \beta(x)u_0(x)}{\alpha(x)}\varphi'(x)\,dx + \int_a^b h(x)\varphi(x)\,dx \tag{3.34}$$

for $\varphi \in H_0^1(a, c)$. With μ_ε defined by $\mu_\varepsilon|_{[a,b]} = \varepsilon$ and $\mu_\varepsilon|_{[b,c]} = \mu$, there follows

$$\tilde{a}(w_\varepsilon, w_\varepsilon) = \int_a^c \mu_\varepsilon(x)(w_\varepsilon'(x))^2\,dx + \int_a^c \left[\beta(x) - \frac{\alpha'(x)}{2}\right] w_\varepsilon^2(x)\,dx.$$

The Poincaré inequality and the property (2.8) imply the coerciveness of $\tilde{a}$ on $H_0^1(a, c)$. Consequently, the problem for $(\tilde{r}_\varepsilon, s_\varepsilon)$ admits a unique solution $w_\varepsilon \in H_0^1(a, c)$, hence $r_\varepsilon \in H^1(a, b)$, $s_\varepsilon \in H^1(b, c)$. Equation (3.12) implies $s_\varepsilon \in H^2(b, c)$ and herewith

$$(r_\varepsilon, s_\varepsilon) \in H^1(a, b) \times H^2(b, c). \tag{3.35}$$

Note that the differential equation in (3.11) is satisfied in the weak sense and the second transmission condition in (3.13) is understood in the sense of traces, which, in fact is the natural flux condition.

Using arguments similar to the preceding proof of (3.35), we conclude that the boundary value problems (3.15)–(3.17), (3.18)–(3.20) and (3.25) are uniquely solvable and that the solutions satisfy

$$(r_1, s_1), (r_2, s_2), (r_3, s_3) \in H^1(a, b) \times H^2(b, c). \tag{3.36}$$

Furthermore, the right-hand sides in (3.26), (3.27) satisfy

$$-\beta r_3 - \beta\ell_\varepsilon \in H^1(a, b) \quad \text{and} \quad -\beta s_3 \in H^2(b, c).$$

Consequently, the arguments used for showing (2.10) can be repeated and we find

$$(r_4, s_4) \in H^2(a, b) \times H^2(b, c). \tag{3.37}$$

4. Uniform estimates of the remainders

In this section we will show that the remainders $(r_\varepsilon, s_\varepsilon)$ converge uniformly to zero as $\varepsilon \to 0$ provided $f \in L^\infty(a, b)$ and continuous at b. This permits us to update *a posteriori* the solution of the degenerate hyperbolic–elliptic boundary-transmission value problem by the boundary layer function ℓ_ε. In section 5 we present corresponding numerical computations.

We now investigate the boundary value problems for the remainder components (r_i, s_i) $(i = 1, \ldots, 4)$ and assume first $f \in C[a, b]$, instead of the weaker assumption $f \in L^\infty(a, b)$. We apply a closure argument in Theorem 4.4 to generalize these results to $f \in L^\infty(a, b)$.

4.1. *Case of vanishing damping* $\beta \equiv 0,\ \alpha' \leqslant 0$

4.1.1. *Boundary value problem* (R_1); *estimates for* (r_1, s_1). The solutions of the differential equations in (3.15), (3.16) can be written in the form

$$r_1(x) = C + C_1 \int_{t=a}^{x} \mathrm{e}^{(1/\varepsilon)\int_a^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t \quad \text{and} \quad s_1(x) = \tilde{C} + \tilde{C}_1 \int_{t=b}^{x} \mathrm{e}^{(1/\mu)\int_b^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t, \tag{4.1}$$

with real constants C, C_1, $\tilde{C}$, $\tilde{C}_1$ depending on ε. The boundary conditions in (3.15) and (3.16) imply

$$C = \mathcal{O}(\varepsilon) \quad \text{and} \quad \tilde{C} + \tilde{C}_1 \int_b^c \mathrm{e}^{(1/\mu)\int_b^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t = 0, \tag{4.2}$$

whereas the transmission conditions (3.17) lead to

$$C + C_1 \int_a^b \mathrm{e}^{(1/\varepsilon)\int_a^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t = \tilde{C} \quad \text{and} \quad \varepsilon C_1 \mathrm{e}^{(1/\varepsilon)\int_a^b \alpha(\tau)\,\mathrm{d}\tau} + \mathcal{O}(\varepsilon) = \mu \tilde{C}_1. \tag{4.3}$$

We denote

$$M := |f|_{L^\infty(a,b)} \quad \text{and} \quad K := \int_b^c \exp\left[\frac{1}{\mu}\int_b^t \alpha(\tau)\,\mathrm{d}\tau\right]\mathrm{d}t. \tag{4.4}$$

With $|\varepsilon u_0'(b)| = \varepsilon|f(b)|/|\alpha(b)| \leqslant \varepsilon|f(b)|/\alpha_0$, there follows $\varepsilon u_0'(b) = \mathcal{O}(\varepsilon)$, where the constant in $\mathcal{O}(\varepsilon)$ depends on $|f(b)|$. From the second relations in (4.2) and (4.3) we get

$$\tilde{C} = -K\tilde{C}_1 \quad \text{and} \quad C_1 = \frac{\mu\tilde{C}_1 - \mathcal{O}(\varepsilon)}{\varepsilon}\,\mathrm{e}^{-(1/\varepsilon)\int_a^b \alpha(\tau)\,\mathrm{d}\tau}. \tag{4.5}$$

Using now the estimate for C and (4.5), we get from the first transmission condition in (4.3) the relation $\mathcal{O}(\varepsilon) + (\mu\tilde{C}_1 - \mathcal{O}(\varepsilon))\cdot\zeta(\varepsilon) = -K\tilde{C}_1$ where $\zeta(\varepsilon) := (1/\varepsilon)\int_a^b \mathrm{e}^{-(1/\varepsilon)\int_{\tau=t}^b \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t \leqslant 1/\varepsilon\int_a^b \mathrm{e}^{-(\alpha_0/\varepsilon)(b-t)}\,\mathrm{d}t \leqslant 1/\alpha_0$ for all $\varepsilon > 0$ due to (2.7). In consequence,

$$\tilde{C}_1 = \mathcal{O}(\varepsilon). \tag{4.6}$$

The relations in (4.5) imply now

$$\tilde{C} = \mathcal{O}(\varepsilon) \quad \text{and} \quad C_1 = \mathcal{O}(\mathrm{e}^{-(1/\varepsilon)\int_a^b \alpha(\tau)\,\mathrm{d}\tau}), \tag{4.7}$$

where, as before, the constants in $\mathcal{O}(\varepsilon)$ and in $\mathcal{O}(\mathrm{e}^{-(1/\varepsilon)\int_a^b \alpha(\tau)\,\mathrm{d}\tau})$ depend on $|f(b)|$.

Lemma 4.1. *Let* $(r_1, s_1) \in H^1(a,b) \times H^2(b,c)$ *be the solution of the coupled boundary value problem* (R_1). *Then, for* $\varepsilon \to 0$, *one has*

$$|r_1|_{C[a,b]} = \mathcal{O}(\varepsilon) \quad \textit{and} \quad |s_1|_{C[b,c]} = \mathcal{O}(\varepsilon). \tag{4.8}$$

Proof. Using (4.1), (4.2) and (4.7), we get

$$r_1(x) = \mathcal{O}(\varepsilon) + \mathcal{O}(\mathrm{e}^{-(1/\varepsilon)\int_a^b \alpha(\tau)\,\mathrm{d}\tau}) \int_a^x \mathrm{e}^{(1/\varepsilon)\int_a^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t \quad \text{for} \quad x \in [a, b].$$

With the aim of (2.7), we obtain, with some constants $c, c_1 > 0$:

$$|r_1(x)| \leqslant c\varepsilon + c_1 \int_a^x \mathrm{e}^{-(1/\varepsilon)\int_t^b \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t \leqslant \left[c + \frac{c_1}{\alpha_0}(\mathrm{e}^{-\alpha_0(b-x)/\varepsilon} - \mathrm{e}^{-\alpha_0(b-a)/\varepsilon}) \right] \cdot \varepsilon$$

for all $x \in [a, b]$.

The first relation in (4.8) is now obvious. With (4.6) and the first estimate in (4.7), we get $|s_1(x)| \leqslant [\tilde{c} + \tilde{c}_1 K]\,\varepsilon$ with some constants $\tilde{c}, \tilde{c}_1 > 0$, the second estimate in (4.8). □

4.1.2. *Boundary value problem* (R_2); *estimates for* (r_2, s_2). The solutions of (3.18) and (3.19) are given by

$$r_2(x) = C_2 \int_a^x \mathrm{e}^{(1/\varepsilon)\int_a^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t - \int_a^x u_0''(t) \cdot \left[\int_t^x \mathrm{e}^{(1/\varepsilon)\int_t^\sigma \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma \right] \mathrm{d}t, \tag{4.9}$$

$$s_2(x) = -\tilde{C}_2 \int_x^c \mathrm{e}^{(1/\mu)\int_b^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t, \tag{4.10}$$

with real constants C_2, $\tilde{C}_2$ depending on ε. In order to get estimates for C_2 and $\tilde{C}_2$, we use the transmission conditions (3.20), which take the form

$$C_2 \int_a^b \mathrm{e}^{(1/\varepsilon)\int_a^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t - \int_a^b u_0''(t) \cdot \left[\int_t^b \mathrm{e}^{(1/\varepsilon)\int_t^\sigma \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma \right] \mathrm{d}t$$
$$= -\tilde{C}_2 \int_b^c \mathrm{e}^{(1/\mu)\int_b^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t, \tag{4.11}$$

$$\varepsilon C_2\,\mathrm{e}^{(1/\varepsilon)\int_a^b \alpha(\tau)\,\mathrm{d}\tau} - \varepsilon \int_a^b u_0''(t)\,\mathrm{e}^{(1/\varepsilon)\int_t^b \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t = \mu \tilde{C}_2. \tag{4.12}$$

Eliminating

$$C_2 = \frac{\mu \tilde{C}_2}{\varepsilon}\,\mathrm{e}^{-(1/\varepsilon)\int_a^b \alpha(\tau)\,\mathrm{d}\tau} + \int_a^b u_0''(t)\,\mathrm{e}^{-(1/\varepsilon)\int_a^t \alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t \tag{4.13}$$

from (4.11), we obtain, with K given by (4.4), the equation

$$h_1(\varepsilon)\,\tilde{C}_2 = h_2(\varepsilon), \tag{4.14}$$

where the functions h_1 and h_2 are given by

$$h_1(\varepsilon) := -K - \frac{\mu}{\varepsilon} e^{-(1/\varepsilon)\int_a^b \alpha(\tau)\,d\tau} \int_a^b e^{(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau}\,dt, \tag{4.15}$$

$$h_2(\varepsilon) := \int_a^b u_0''(t)\,e^{-(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau}\,dt \int_a^b e^{(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau}\,dt - \int_a^b u_0''(t)\left[\int_t^b e^{(1/\varepsilon)\int_t^\sigma \alpha(\tau)\,d\tau}\,d\sigma\right]dt. \tag{4.16}$$

Lemma 4.2. *The constant* $\tilde{C}_2$ *satisfies*

$$\tilde{C}_2 = \mathcal{O}(\varepsilon), \tag{4.17}$$

where the constant in $\mathcal{O}(\varepsilon)$ *depends on* M *and* $|f(b)|$.

Proof. We split the proof into four steps. We use (4.14) and investigate the asymptotic behavior of h_1 and h_2.

Step 1. The function h_1 from (4.15) satisfies

$$-K - \frac{\mu}{\alpha_0} \leqslant h_1(\varepsilon) < -K \quad \text{for all} \quad \varepsilon > 0. \tag{4.18}$$

The second inequality is evident. We also have $|h_1(\varepsilon)| \leqslant K + (\mu/\varepsilon)\int_a^b e^{-(1/\varepsilon)\int_t^b \alpha(\tau)\,d\tau}\,dt$. Using now (2.7), we get the estimate

$$|h_1(\varepsilon)| \leqslant K + \frac{\mu}{\varepsilon}\int_a^b e^{-(\alpha_0/\varepsilon)(b-t)}\,dt \leqslant K + \frac{\mu}{\alpha_0}.$$

Step 2. The function h_2 can be represented in the form

$$h_2(\varepsilon) = \int_a^b u_0''(t)\left[\int_a^t e^{-(1/\varepsilon)\int_\sigma^t \alpha(\tau)\,d\tau}\,d\sigma\right]dt \quad \text{for} \quad \varepsilon > 0. \tag{4.19}$$

Indeed, the first term in the definition of h_2 can be written as

$$\int_{t=a}^b u_0''(t)\left[\int_{\sigma=a}^b e^{(1/\varepsilon)\int_{\tau=a}^\sigma \alpha(\tau)\,d\tau}\,d\sigma\right] e^{-(1/\varepsilon)\int_{\tau=a}^t \alpha(\tau)\,d\tau}\,dt$$

$$= \int_{t=a}^b u_0''(t)\left[\int_{\sigma=a}^t e^{(1/\varepsilon)\int_{\tau=a}^\sigma \alpha(\tau)\,d\tau} \cdot e^{-(1/\varepsilon)\int_{\tau=a}^t \alpha(\tau)\,d\tau}\,d\sigma\right]dt$$

$$+ \int_{t=a}^b u_0''(t)\left[\int_{\sigma=t}^b e^{(1/\varepsilon)\int_{\tau=a}^\sigma \alpha(\tau)\,d\tau} \cdot e^{-(1/\varepsilon)\int_{\tau=a}^t \alpha(\tau)\,d\tau}\,d\sigma\right]dt.$$

Since within the second integral $a \leqslant t \leqslant \sigma \leqslant b$, it can be written as

$$\int_a^b u_0''(t)\left[\int_t^b e^{(1/\varepsilon)\int_t^\sigma \alpha(\tau)\,d\tau}\,d\sigma\right]dt$$

and, hence, it is the last integral in (4.16). Within the first integral, $a \leqslant \sigma \leqslant t \leqslant b$; and the integrand of the interior integral is $\exp[-(1/\varepsilon)\int_{\tau=\sigma}^{t}\alpha(\tau)\,d\tau]$, which yields (4.19). Integration by parts in (4.19) leads to

$$h_2(\varepsilon) = u_0'(b)\int_a^b e^{-(1/\varepsilon)\int_\sigma^b \alpha(\tau)\,d\tau}\,d\sigma - \int_a^b u_0'(t)\left[1 - \frac{\alpha(t)}{\varepsilon}\int_a^t e^{-(1/\varepsilon)\int_\sigma^t \alpha(\tau)\,d\tau}\,d\sigma\right]dt. \tag{4.20}$$

Step 3. We now use $|f(b)| \leqslant M$, $\alpha(\tau) \geqslant \alpha_0 > 0$ as well as $\alpha(b)u_0'(b) = f(b)$, in order to estimate the first expression in (4.20). We get

$$\left|u_0'(b)\int_a^b e^{-(1/\varepsilon)\int_\sigma^b \alpha(\tau)\,d\tau}\,d\sigma\right| \leqslant |u_0'(b)|\int_a^b e^{-(\alpha_0/\varepsilon)(b-\sigma)}\,d\sigma$$

$$= \frac{M}{\alpha_0^2}(1 - e^{-\alpha_0(b-a)/\varepsilon})\varepsilon \leqslant \frac{M}{\alpha_0^2}\varepsilon,$$

and, consequently,

$$u_0'(b)\int_a^b e^{-(1/\varepsilon)\int_\sigma^b \alpha(\tau)\,d\tau}\,d\sigma = \mathcal{O}(\varepsilon). \tag{4.21}$$

Note that the constant in $\mathcal{O}(\varepsilon)$ depends on $|f(b)|$. The second term in (4.20) can be reformulated as

$$\int_a^b u_0'(t)\left[\int_a^t \frac{\alpha(t) - \alpha(\sigma) + \alpha(\sigma)}{\varepsilon} e^{-(1/\varepsilon)\int_\sigma^t \alpha(\tau)\,d\tau}\,d\sigma - 1\right]dt =: h_{2,1}(\varepsilon) + h_{2,2}(\varepsilon)$$

with

$$h_{2,1}(\varepsilon) := \int_a^b u_0'(t)\left[\int_a^t \frac{\alpha(t) - \alpha(\sigma)}{\varepsilon} e^{-(1/\varepsilon)\int_\sigma^t \alpha(\tau)\,d\tau}\,d\sigma\right]dt, \tag{4.22}$$

$$h_{2,2}(\varepsilon) := \int_a^b u_0'(t)\left[\int_a^t \frac{\alpha(\sigma)}{\varepsilon} e^{-(1/\varepsilon)\int_\sigma^t \alpha(\tau)\,d\tau}\,d\sigma - 1\right]dt. \tag{4.23}$$

Step 4: The functions $h_{2,1}$ and $h_{2,2}$ satisfy

$$h_{2,1}(\varepsilon) = \mathcal{O}(\varepsilon) \quad \text{and} \quad h_{2,2}(\varepsilon) = \mathcal{O}(\varepsilon), \tag{4.24}$$

where the constants in $\mathcal{O}(\varepsilon)$ depend on M only. In order to show (4.24), we first use the boundedness of $u_0'(t) = f(t)/\alpha(t)$ on $[a, b]$ ($|u_0'(t)| \leqslant M/\alpha_0$ on $[a, b]$), and then take into account that α is Lipschitz continuous and satisfies (2.7). We get

$$|h_{2,1}(\varepsilon)| \leqslant \frac{M}{\alpha_0}\int_a^b\int_a^t \frac{L(t-\sigma)}{\varepsilon} e^{-(\alpha_0/\varepsilon)(t-\sigma)}\,d\sigma\,dt$$

$$\leqslant \frac{ML}{\alpha_0^2}\int_a^b\left[\frac{\varepsilon}{\alpha_0} - \left(t - a + \frac{\varepsilon}{\alpha_0}\right)e^{-\alpha_0(t-a)/\varepsilon}\right]dt \leqslant \tilde{M}(\varepsilon)\varepsilon$$

with

$$\tilde{M}(\varepsilon) := \frac{ML}{\alpha_0^2}\left[\frac{b-a}{\alpha_0} - 2\frac{\varepsilon}{\alpha_0^2} + \left((b-a)\frac{1}{\alpha_0} + 2\frac{\varepsilon}{\alpha_0^2}\right)e^{-\alpha_0(b-a)/\varepsilon}\right]$$

which is uniformly bounded for $\varepsilon > 0$. Herewith the first estimate in (4.24) is proved.

With

$$\frac{\alpha(\sigma)}{\varepsilon} e^{-(1/\varepsilon)\int_\sigma^t \alpha(\tau)\,d\tau} = \frac{d}{d\sigma}\left[e^{-(1/\varepsilon)\int_\sigma^t \alpha(\tau)\,d\tau}\right],$$

we get

$$h_{2,2}(\varepsilon) = -\int_a^b u_0'(t)\exp\left[-\frac{1}{\varepsilon}\int_a^t \alpha(\tau)\,d\tau\right]dt.$$

Using the relation $|u_0'(t)| \leqslant M/\alpha_0$ in $[a, b]$, we obtain the second estimate in (4.24):

$$|h_{2,2}(\varepsilon)| \leqslant \frac{M}{\alpha_0}\int_a^b e^{-(\alpha_0/\varepsilon)(t-a)}\,dt \leqslant \frac{M}{\alpha_0^2}\,\varepsilon.$$

Finally, (4.21) and (4.24) imply the existence of a constant $\bar{c} > 0$, such that

$$|h_2(\varepsilon)| \leqslant \bar{c}\varepsilon \quad \text{for all} \quad \varepsilon > 0. \tag{4.25}$$

The estimate (4.17) now follows from (4.14), (4.18) and (4.25). The lemma is proved. □

In order to get an uniform estimate for r_2 in (4.9), we insert (4.17) into (4.13) and obtain

$$C_2 = \mu\mathcal{O}(1)\,e^{-(1/\varepsilon)\int_a^b \alpha(\tau)\,d\tau} + \int_a^b u_0''(t)\,e^{-(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau}\,dt.$$

Inserting C_2 into (4.9), we get

$$\begin{aligned} r_2(x) = {} & \underbrace{\mu\mathcal{O}(1)\,e^{-(1/\varepsilon)\int_a^b \alpha(\tau)\,d\tau}\int_a^x e^{(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau}\,dt}_{=: r_{2,1}(x)} \\ & \underbrace{+\left(\int_a^b u_0''(t)\,e^{-(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau}\,dt\right)\int_a^x e^{(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau}\,dt - \int_a^x u_0''(t)\left[\int_t^x e^{(1/\varepsilon)\int_t^\sigma \alpha(\tau)\,d\tau}\,d\sigma\right]dt}_{=: r_{2,2}(x)}. \end{aligned} \tag{4.26}$$

Theorem 4.3. *The solution* $(r_2, s_2) \in H^1(a, b) \times H^2(b, c)$ *of the problem* (R_2) *satisfies*

$$|r_2|_{C[a,b]} = \mathcal{O}(\varepsilon) \quad \text{and} \quad |s_2|_{C[b,c]} = \mathcal{O}(\varepsilon) \quad \text{for} \quad \varepsilon \to 0. \tag{4.27}$$

Here the constants in $\mathcal{O}(\varepsilon)$ *depend on* M *and* $|f(b)|$.

Proof. We first establish estimates for $r_{2,j}$. For $r_{2,1}$ one obtains for all $x \in [a, b]$:

$$|r_{2,1}(x)| = c \left| \int_a^x e^{-(1/\varepsilon)\int_t^b \alpha(\tau)\,d\tau}\,dt \right| \leqslant c \int_a^x e^{-(\alpha_0/\varepsilon)(b-t)}\,dt$$

$$= \frac{c}{\alpha_0}\left(e^{-\alpha_0(b-x)/\varepsilon} - e^{-\alpha_0(b-a)/\varepsilon}\right)\varepsilon,$$

with a generic constant $c > 0$, and, consequently,

$$|r_{2,1}|_{C[a,b]} = \mathcal{O}(\varepsilon). \tag{4.28}$$

Furthermore, we shall prove the relation

$$r_{2,2}(x) = \int_{t=a}^x \left[\int_{\sigma=t}^b u_0''(\sigma)\, e^{-(1/\varepsilon)\int_{\tau=t}^{\sigma}\alpha(\tau)\,d\tau}\,d\sigma \right] dt \quad \text{for} \quad x \in [a, b]. \tag{4.29}$$

Indeed, inserting the constant $\int_a^b u_0''(t) \exp[-(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau]\,dt$ into the integral $\int_a^x \exp[(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau]\,dt$, we get for the first term $\tilde{r}_{2,2}$ of $r_{2,2}$ in (4.26) the expression

$$\tilde{r}_{2,2}(x) = \int_{t=a}^x \left[\int_{\sigma=a}^b u_0''(\sigma)\, e^{-(1/\varepsilon)\int_a^{\sigma}\alpha(\tau)\,d\tau}\,d\sigma \right] e^{(1/\varepsilon)\int_a^t \alpha(\tau)\,d\tau}\,dt$$

$$= \int_{t=a}^x \left[\int_{\sigma=a}^t u_0''(\sigma)\, e^{(1/\varepsilon)\int_{\tau=\sigma}^t \alpha(\tau)\,d\tau}\,d\sigma \right] dt$$

$$+ \int_{t=a}^x \left[\int_{\sigma=t}^b u_0''(\sigma)\, e^{-(1/\varepsilon)\int_{\tau=t}^{\sigma}\alpha(\tau)\,d\tau}\,d\sigma \right] dt.$$

By changing the order of integration, the first integral in the last relation can be reformulated as

$$\int_{\sigma=a}^x \left[\int_{t=\sigma}^x u_0''(\sigma)\, e^{(1/\varepsilon)\int_{\tau=\sigma}^t \alpha(\tau)\,d\tau}\,dt \right] d\sigma \overset{t\leftrightarrow\sigma}{=} \int_{t=a}^x u_0''(t) \left[\int_{\sigma=t}^x e^{(1/\varepsilon)\int_{\tau=t}^{\sigma}\alpha(\tau)\,d\tau}\,d\sigma \right] dt$$

and equals the last integral in the expression of $r_{2,2}(x)$ in (4.26). The second integral coincides with that from (4.29). Now, it can be shown that $r_{2,2}$ satisfies

$$|r_{2,2}|_{C[a,b]} = \mathcal{O}(\varepsilon), \tag{4.30}$$

where the constant in $\mathcal{O}(\varepsilon)$ depends on M. For the sake of readability, the corresponding proof is given in Appendix C.

The estimates (4.28) and (4.30), together with the representation (4.26), imply the desired estimate for r_2 in (4.27).

For the estimate of $|s_2(x)|$ note that the second relation in (4.27) is a direct consequence of (4.10), together with the inequality

$$|s_2(x)| = |\tilde{C}_2| \left| -\int_x^c e^{(1/\mu)\int_b^t \alpha(\tau)\,d\tau}\, dt \right| \leqslant |\tilde{C}_2| K,$$

where K is given by (4.4), and with (4.17). This completes the proof of the theorem. □

Using Lemma 4.1 and Theorem 4.3, we now formulate:

Theorem 4.4. *Let $f \in L^\infty(a, b)$ and f be continuous at b; let $g \in L^2(b, c)$. Further let $(u_\varepsilon, v_\varepsilon) \in H^2(a, b) \times H^2(b, c)$ be the solution of the elliptic–elliptic boundary value problem with $\varepsilon > 0$ and let $(u_0, v_0) \in H^1(a, b) \times H^2(b, c)$ be the solution of the degenerate hyperbolic–elliptic boundary value problem* (P_0). *Then, as $\varepsilon \to 0$, the remainders $(r_\varepsilon, s_\varepsilon)$ satisfy*

$$|r_\varepsilon|_{C[a,b]} = |u_\varepsilon - u_0 - \ell_\varepsilon|_{C[a,b]} = \mathcal{O}(\varepsilon) \quad \textit{and} \quad |s_\varepsilon|_{C[b,c]} = |v_\varepsilon - v_0|_{C[b,c]} = \mathcal{O}(\varepsilon), \tag{4.31}$$

where the constants in $\mathcal{O}(\varepsilon)$ depend on $|f(b)|$ and M.

Proof. The unique solvability of the problems (P_ε) and (P_0) is clear, see section 2.

For $f \in C[a, b]$, the estimates (4.31) follow from the representation (3.14), together with the estimates in (4.8) and (4.27).

Let us now consider $f \in L^\infty(a, b)$ and approximate f by a sequence $\{f_n\} \subset C[a, b]$:

$$\| f - f_n \|_{L^2(a,b)} \to 0 \tag{4.32}$$

with $|f_n(x)| \leqslant 2M$ and $f_n(b) = f(b)$. Associated with $\{f_n\}$, we consider the sequence $(u_{0,n}, v_{0,n})$ of solutions to the problems

$$\alpha(x) u'_{0,n}(x) = f_n(x) \quad \text{for} \quad x \in (a, b), \qquad u_{0,n}(a) = 0,$$

$$-\mu v''_{0,n}(x) + \alpha(x) v'_{0,n}(x) = g(x) \quad \text{for} \quad x \in (b, c), \quad v_{0,n}(c) = 0,$$

$$-\mu v'_{0,n}(b) + \alpha(b) v_{0,n}(b) = \alpha(b) u_{0,n}(b).$$

Furthermore, we denote by $(r_{\varepsilon,n}, s_{\varepsilon,n})$ the solutions of the boundary value problem (3.11)–(3.13) with $u_{0,n}$ instead u_0. Due to our preliminary results for continuous right-hand sides,

$$|r_{\varepsilon,n}|_{C[a,b]} = \mathcal{O}(\varepsilon) \quad \text{and} \quad |s_{\varepsilon,n}|_{C[b,c]} = \mathcal{O}(\varepsilon) \quad \text{for all} \quad n \in \mathbb{N} \tag{4.33}$$

holds.

Note that the constants in $\mathcal{O}(\varepsilon)$ now depend only on M and $f_n(b) = f(b)$, but not on n. For $f \in L^2(a, b)$ but $f \notin L^\infty(a, b)$, these constants would depend on n and may not be uniformly bounded. With the solution $(r_\varepsilon, s_\varepsilon) \in H^1(a, b) \times H^2(b, c)$ of the boundary value problem (3.11)–(3.13)—which has to be estimated here—we get that the pair of functions $(\varrho_{\varepsilon,n}, \sigma_{\varepsilon,n}) := (r_\varepsilon - r_{\varepsilon,n}, s_\varepsilon - s_{\varepsilon,n})$ solves the following coupled problem:

$$-\varepsilon\varrho''_{\varepsilon,n}(x) + \alpha(x)\varrho'_{\varepsilon,n}(x) = \varepsilon(u_0 - u_{0,n})''(x)$$
$$\text{for} \quad x \in (a, b), \quad \varrho_{\varepsilon,n}(a) = \Omega_n(\varepsilon), \tag{4.34}$$

$$(\delta \mathrm{R}_\varepsilon): \quad -\mu\sigma''_{\varepsilon,n}(x) + \alpha(x)\sigma'_{\varepsilon,n}(x) = 0 \quad \text{for} \quad x \in (b, c), \quad \sigma_{\varepsilon,n}(c) = 0, \tag{4.35}$$

$$\varrho_{\varepsilon,n}(b) = \sigma_{\varepsilon,n}(b), \quad \varepsilon\varrho'_{\varepsilon,n}(b) + \varepsilon(u_0 - u_{0,n})'(b) = \mu\sigma'_{\varepsilon,n}(b). \tag{4.36}$$

Here,

$$\Omega_n(\varepsilon) := r_\varepsilon(a) - r_{\varepsilon,n}(a) = -(\Sigma_0 - \Sigma_{0,n})\left[1 - \alpha(b)\cdot\int_0^{(b-a)/\varepsilon} \mathrm{e}^{-\int_0^s \alpha(b-\varepsilon u)\,\mathrm{d}u}\,\mathrm{d}s\right],$$

where $\Sigma_0 := v_0(b) - u_0(b)$ and $\Sigma_{0,n} := v_{0,n}(b) - u_{0,n}(b)$ (see (3.10)). The main task for the proof is to find an uniform estimate for the quantity $(r_\varepsilon - r_{\varepsilon,n}, s_\varepsilon - s_{\varepsilon,n})$. Such an estimate, coupled with (4.33), would lead to the desired result, i.e. estimates for $(r_\varepsilon, s_\varepsilon)$.

Multiplying the equations in (4.34) and (4.35) by $\varrho_{\varepsilon,n}$ and $\sigma_{\varepsilon,n}$, respectively, integrating by parts and using the boundary and transmission conditions, we obtain

$$\begin{aligned}
&\varepsilon\,\|\varrho'_{\varepsilon,n}\|^2_{L^2(a,b)} + \mu\,\|\sigma'_{\varepsilon,n}\|^2_{L^2(b,c)} + \int_a^b [-\alpha'(x)/2]\,\varrho^2_{\varepsilon,n}(x)\,\mathrm{d}x \\
&\quad + \int_b^c [-\alpha'(x)/2]\,\sigma^2_{\varepsilon,n}(x)\,\mathrm{d}x \\
&= -\varepsilon\int_a^b (u'_0 - u'_{0,n})\,\varrho'_{\varepsilon,n}\,\mathrm{d}x + \frac{\alpha(a)}{2}\,\varrho^2_{\varepsilon,n}(a) - \varepsilon(r'_\varepsilon - r'_{\varepsilon,n})(a)\,\varrho_{\varepsilon,n}(a) \\
&\quad - \varepsilon(u'_0 - u'_{0,n})(a)\,\varrho_{\varepsilon,n}(a) \\
&= -\varepsilon\int_a^b \frac{(f - f_n)(x)}{\alpha(x)}\,\varrho'_{\varepsilon,n}(x)\,\mathrm{d}x + \frac{\alpha(a)}{2}\,\varrho^2_{\varepsilon,n}(a) \\
&\quad + \left\{\alpha(b)(\Sigma_0 - \Sigma_{0,n})\exp\left[-\int_0^{(b-a)/\varepsilon} \alpha(b - \varepsilon u)\,\mathrm{d}u\right]\right. \\
&\quad \left. - \varepsilon(u'_\varepsilon(a) - u'_{\varepsilon,n}(a))\right\}\varrho_{\varepsilon,n}(a).
\end{aligned} \tag{4.37}$$

We first estimate the right-hand side. First we have

$$|u_0(b) - u_{0,n}(b)| = \left|\int_a^b \frac{(f - f_n)(t)}{\alpha(t)}\,\mathrm{d}t\right| \leqslant \frac{1}{\alpha_0}\sqrt{b-a}\,\|f - f_n\|_{L^2(a,b)}.$$

Multiplying the equation $-\mu(v''_0 - v''_{0,n})(x) + \alpha(x)(v'_0 - v'_{0,n})(x) = 0$ by $v_0 - v_{0,n}$, integrating over $[b, c]$ and using the transmission condition $-\mu(v'_0 - v'_{0,n})(b) + \alpha(b)(v_0 - v_{0,n})(b) = \alpha(b)(u_0 - u_{0,n})(b)$ we get

$$|v_0(b) - v_{0,n}(b)| \leqslant 2\,|u_0(b) - u_{0,n}(b)|,$$

hence,

$$|\Sigma_0 - \Sigma_{0,n}| \leqslant \frac{3}{\alpha_0}\sqrt{b-a}\, \| f - f_n \|_{L^2(a,b)}.$$

Consequently,

$$|r_\varepsilon(a) - r_{\varepsilon,n}(a)| = |\Omega_n(\varepsilon)| \leqslant \frac{3}{\alpha_0}\sqrt{b-a}\, |\tilde{\omega}(\varepsilon)| \cdot \| f - f_n \|_{L^2(a,b)}, \tag{4.38}$$

with the function

$$\tilde{\omega}(\varepsilon) := 1 - \alpha(b) \int_0^{(b-a)/\varepsilon} \exp\left[-\int_0^s \alpha(b - \varepsilon u)\, \mathrm{d}u \right] \mathrm{d}s = \mathcal{O}(\varepsilon), \tag{4.39}$$

as will be shown in Appendix A. We proceed with our analysis by estimating the quantity $-\varepsilon(u'_\varepsilon(a) - u'_{\varepsilon,n}(a))$ appearing on the right-hand side of (4.37). For this purpose, consider the transmission-boundary value problem for $(\xi_{\varepsilon,n}, \eta_{\varepsilon,n}) := (u_\varepsilon - u_{\varepsilon,n}, v_\varepsilon - v_{\varepsilon,n})$:

$$-\varepsilon \xi''_{\varepsilon,n}(x) + \alpha(x) \xi'_{\varepsilon,n}(x) = (f - f_n)(x) \quad \text{for} \quad x \in (a,b), \quad \xi_{\varepsilon,n}(a) = 0, \tag{4.40}$$

$$(\delta \mathrm{P}_\varepsilon): \quad -\mu \eta''_{\varepsilon,n}(x) + \alpha(x) \eta'_{\varepsilon,n}(x) = 0 \quad \text{for} \quad x \in (b,c), \quad \eta_{\varepsilon,n}(c) = 0, \tag{4.41}$$

$$\xi_{\varepsilon,n}(b) = \eta_{\varepsilon,n}(b), \quad \varepsilon \xi'_{\varepsilon,n}(b) = \mu \eta'_{\varepsilon,n}(b). \tag{4.42}$$

By integrating (4.40) twice and by using $\xi_{\varepsilon,n}(a) = 0$, we get for arbitrary $x \in [a,b]$:

$$-\varepsilon \xi_{\varepsilon,n}(x) + (x-a)\varepsilon \xi'_{\varepsilon,n}(a) + \int_a^x \int_a^t \alpha(\tau) \xi'_{\varepsilon,n}(\tau)\, \mathrm{d}\tau\, \mathrm{d}t = \int_a^x \int_a^t (f - f_n)(\tau)\, \mathrm{d}\tau\, \mathrm{d}t.$$

Integration over $[a,b]$ leads to

$$\varepsilon \xi'_{\varepsilon,n}(a) = \frac{2}{(b-a)^2} \cdot \left[\varepsilon \int_a^b \xi_{\varepsilon,n}(x)\, \mathrm{d}x - \int_a^b \int_a^x \int_a^t \alpha(\tau) \xi'_{\varepsilon,n}(\tau)\, \mathrm{d}\tau\, \mathrm{d}t\, \mathrm{d}x + \int_a^b \int_a^x \int_a^t (f - f_n)(\tau)\, \mathrm{d}\tau\, \mathrm{d}t\, \mathrm{d}x \right].$$

The Schwarz inequality implies

$$|\varepsilon(u'_\varepsilon(a) - u'_{\varepsilon,n}(a))| \leqslant (c_1 \varepsilon + c_2) \| u_\varepsilon - u_{\varepsilon,n} \|_{L^2(a,b)} + c_3 \| f - f_n \|_{L^2(a,b)}, \tag{4.43}$$

with some real constants $c_1, c_2, c_3 > 0$, depending on $\| \alpha \|_{L^2(a,b)}$ and on $\| \alpha' \|_{L^2(a,b)}$. The boundary value problem $(\delta \mathrm{P}_\varepsilon)$ has the same form as the original problem (2.1)–(2.3). Beyond existence and uniqueness of the solution $\zeta_{\varepsilon,n} \in H_0^1(a,c)$ with $\zeta_{\varepsilon,n}|_{[a,b]} = \xi_{\varepsilon,n} \in H^2(a,b)$ and $\zeta_{\varepsilon,n}|_{[b,c]} = \eta_{\varepsilon,n} \in H^2(b,c)$, the Lax–Milgram theorem yields also

$$\| u_\varepsilon - u_{\varepsilon,n} \|_{L^2(a,b)} = \| \xi_{\varepsilon,n} \|_{L^2(a,b)} \leqslant \| \zeta_{\varepsilon,n} \|_{H^1(a,c)} \leqslant \frac{1}{\gamma_0(\varepsilon)} \| f - f_n \|_{L^2(a,b)}, \tag{4.44}$$

where $\gamma_0(\varepsilon)$ is the coerciveness constant from (2.9). Inserting the last estimate into (4.43), we obtain

$$|\varepsilon(u'_\varepsilon(a) - u'_{\varepsilon,n}(a))| \leqslant c_4(\varepsilon) \| f - f_n \|_{L^2(a,b)}, \tag{4.45}$$

with some constant $c_4(\varepsilon)$ depending on c_1, c_2, c_3 and on $\gamma_0(\varepsilon)$. Thus, the right-hand side of (4.37) can completely be estimated.

With the Schwarz inequality, the estimates (4.38), (4.45), as well as the properties (2.7) and $-\alpha'(x) \geqslant 0$ for $x \in [a, c]$, we derive from (4.37) the inequality

$$\| \varrho'_{\varepsilon,n} \|^2_{L^2(a,b)} + \frac{\mu}{\varepsilon} \| \sigma'_{\varepsilon,n} \|^2_{L^2(b,c)} \leqslant \frac{1}{\alpha_0} \| f - f_n \|_{L^2(a,b)} \cdot \| \varrho'_{\varepsilon,n} \|_{L^2(a,b)} + c_5(\varepsilon) \| f - f_n \|^2_{L^2(a,b)} \tag{4.46}$$

with

$$c_5(\varepsilon) := \frac{9\alpha(a)(b-a)}{2\varepsilon\alpha_0^2} \tilde{\omega}^2(\varepsilon) + \frac{9\alpha(b)(b-a)}{\varepsilon\alpha_0^2} \exp\left[- \int_0^{(b-a)/\varepsilon} \alpha(b - \varepsilon u)\, \mathrm{d}u \right] \cdot \tilde{\omega}(\varepsilon) + \frac{3c_4(\varepsilon)\sqrt{b-a}}{\varepsilon\alpha_0} \tilde{\omega}(\varepsilon).$$

Due to (4.39), $c_5(\varepsilon)$ is uniformly bounded for $\varepsilon > 0$. Consequently,

$$\| r'_\varepsilon - r'_{\varepsilon,n} \|_{L^2(a,b)} = \| \varrho'_{\varepsilon,n} \|_{L^2(a,b)} \leqslant K(\varepsilon) \| f - f_n \|_{L^2(a,b)}, \tag{4.47}$$

with $K(\varepsilon) := 1/(2\alpha_0) + \frac{1}{2}\sqrt{1/\alpha_0^2 + 4c_5(\varepsilon)}$ which is bounded for $\varepsilon \to 0$. Using the mean value theorem and the Schwarz inequality, we obtain for all $x \in [a, b]$,

$$|r_\varepsilon(x) - r_{\varepsilon,n}(x)| \leqslant \left[\frac{3\sqrt{b-a}}{\alpha_0} |\tilde{\omega}(\varepsilon)| + K(\varepsilon)\sqrt{b-a} \right] \| f - f_n \|_{L^2(a,b)}.$$

Taking into account the convergence in (4.32), for n large enough, we get the inequality

$$|r_\varepsilon(x) - r_{\varepsilon,n}(x)| < \varepsilon \quad \text{for} \quad x \in [a, b]. \tag{4.48}$$

The first relation in (4.31) now follows from the first estimate in (4.33) together with (4.48). Then (4.46) and (4.47) imply

$$\mu \| s'_\varepsilon - s'_{\varepsilon,n} \|^2_{L^2(b,c)} \leqslant \varepsilon \left[\frac{K(\varepsilon)}{\alpha_0} + c_5(\varepsilon) \right] \cdot \| f - f_n \|^2_{L^2(a,b)}.$$

Taking into account the boundary conditions $s_\varepsilon(c) = s_{\varepsilon,n}(c) = 0$ in (3.12) and the convergence in (4.32) one gets

$$|s_\varepsilon - s_{\varepsilon,n}|_{C[b,c]} = \mathcal{O}(\varepsilon),$$

provided n is large enough. This relation, together with the second estimate in (4.33) implies the second estimate in (4.31). This completes the proof of the theorem. □

4.2. *Case of non-vanishing damping* $\beta \geqslant 0, \beta - \frac{1}{2}\alpha' \geqslant 0$

4.2.1. *Boundary value problem* (R_3); *estimates for* (r_3, s_3). The relations (3.25) of the problem (R_3) are *identical* to those of the problem (3.11)–(3.13) for the case $\beta \equiv 0$, $\alpha' \leqslant 0$. Hence, Theorem 4.4 provides us with uniform estimates as follows:

Lemma 4.5. *For $f \in L^\infty(a, b)$ and f continuous at b, the unique solution $(r_3, s_3) \in H^1(a, b) \times H^2(b, c)$ of the boundary value problem* (R_3) *satisfies for $\varepsilon \to 0$ the estimates*

$$|r_3|_{C[a,b]} = \mathcal{O}(\varepsilon) \quad \text{and} \quad |s_3|_{C[b,c]} = \mathcal{O}(\varepsilon). \tag{4.49}$$

Here, the constants in $\mathcal{O}(\varepsilon)$ depend on $|f(b)|$ and M.

4.2.2. *Boundary value problem* (R_4); *estimates for* (r_4, s_4). Note that the formulation of the boundary value problem (3.26)–(3.28) does not contain f, and therefore no auxiliary assumptions on f are necessary.

Let us multiply (3.26) by r_4 and integrate over $[a, b]$. Then, by using the boundary condition $r_4(a) = 0$ and integrating by parts, we get

$$\varepsilon \int_a^b (r_4'(x))^2 \, dx + \int_a^b \left[\beta(x) - \frac{1}{2}\alpha'(x)\right] r_4^2(x) \, dx - \varepsilon r_4'(b) r_4(b) + \frac{1}{2}\alpha(b) r_4^2(b)$$

$$= -\int_a^b \beta[r_3 + \ell_\varepsilon] r_4 \, dx.$$

Multiplying (3.27) by $s_4(x)$ and integrating by parts over $[b, c]$, we obtain with $s_4(c) = 0$,

$$\mu \int_b^c (s_4'(x))^2 \, dx + \int_b^c \left[\beta(x) - \frac{1}{2}\alpha'(x)\right] s_4^2(x) \, dx + \mu s_4'(b) s_4(b) - \frac{1}{2}\alpha(b) s_4^2(b)$$

$$= -\int_b^c \beta s_3 s_4 \, dx.$$

Now use the transmission conditions (3.28) and find by summation

$$\begin{aligned} &\varepsilon \int_a^b (r_4'(x))^2 \, dx + \mu \int_b^c (s_4'(x))^2 dx + \int_a^b \left[\beta(x) - \frac{1}{2}\alpha'(x)\right] r_4^2(x) \, dx \\ &\quad + \int_b^c \left[\beta(x) - \frac{1}{2}\alpha'(x)\right] s_4^2(x) \, dx \\ &= -\int_a^b \beta(x)[r_3(x) + \ell_\varepsilon(x)] \, r_4(x) \, dx - \int_b^c \beta(x) s_3(x) s_4(x) \, dx. \end{aligned} \tag{4.50}$$

For the right-hand side in (4.50), the estimates in (4.49) imply

$$\|r_3\|_{L^2(a,b)} = \mathcal{O}(\varepsilon) \quad \text{and} \quad \|s_3\|_{L^2(b,c)} = \mathcal{O}(\varepsilon). \tag{4.51}$$

The boundary layer function ℓ_ε satisfies $\|\ell_\varepsilon\|_{L^2(a,b)} = \mathcal{O}(\sqrt{\varepsilon})$ (relation (3.21)). Using $\beta \in L^\infty(a, c)$ and the Schwarz inequality, we get

$$|\text{r.h.s.}| \leqslant \|\beta\|_{L^\infty(a,b)} \|r_3 + \ell_\varepsilon\|_{L^2(a,b)} \|r_4\|_{L^2(a,b)} + \|\beta\|_{L^\infty(b,c)} \|s_3\|_{L^2(b,c)} \|s_4\|_{L^2(b,c)}$$
$$\leqslant \mathcal{O}(\sqrt{\varepsilon}) [\|r_4\|_{L^2(a,b)} + \|s_4\|_{L^2(b,c)}]. \tag{4.52}$$

Via r_3 and s_3, the constants in $\mathcal{O}(\sqrt{\varepsilon})$ depend on $|f(b)|$ and M.

In order to proceed we have to make a new assumption on α and β, which is stronger than the relation (2.8); the latter was necessary and sufficient for existence and uniqueness of the solution. We now assume that there exists a real constant ϑ_0 such that

$$\beta(x) - \frac{1}{2}\alpha'(x) \geqslant \vartheta_0 > 0 \quad \text{for almost all} \quad x \in [a, c]. \tag{4.53}$$

With this assumption, we now get an estimate for the left-hand side in (4.50):

$$|\text{l.h.s.}| \geqslant \vartheta_0 [\|r_4\|^2_{L^2(a,b)} + \|s_4\|^2_{L^2(b,c)}] \geqslant \frac{\vartheta_0}{2} [\|r_4\|_{L^2(a,b)} + \|s_4\|_{L^2(b,c)}]^2. \tag{4.54}$$

From the relations (4.52) and (4.54) we obtain $\|r_4\|_{L^2(a,b)} + \|s_4\|_{L^2(b,c)} = \mathcal{O}(\sqrt{\varepsilon})$, hence,

$$\|r_4\|_{L^2(a,b)} = \mathcal{O}(\sqrt{\varepsilon}), \qquad \|s_4\|_{L^2(b,c)} = \mathcal{O}(\sqrt{\varepsilon}). \tag{4.55}$$

Using the estimates in (4.55) as well as the estimate (4.52), we get from (4.50)

$$\varepsilon \int_a^b (r_4'(x))^2 \, \mathrm{d}x + \mu \int_b^c (s_4'(x))^2 \, \mathrm{d}x \leqslant \mathcal{O}(\sqrt{\varepsilon}) [\|r_4\|_{L^2(a,b)} + \|s_4\|_{L^2(b,c)}] = \mathcal{O}(\varepsilon),$$

and, consequently,

$$\|r_4'\|_{L^2(a,b)} \quad \text{is uniformly bounded} \quad \text{and} \quad \|s_4'\|_{L^2(b,c)} = \mathcal{O}(\sqrt{\varepsilon}). \tag{4.56}$$

With these preliminary results, we are now able to state uniform estimates for (r_4, s_4):

Theorem 4.6. *Assume that the coefficients α and β satisfy* (4.53). *Then the solution* $(r_4, s_4) \in H^2(a, b) \times H^2(b, c)$ *of the problem* (R_4) *satisfies*

$$|r_4|_{C[a,b]} = \mathcal{O}(\sqrt{\varepsilon}) \quad \text{and} \quad |s_4|_{C[b,c]} = \mathcal{O}(\sqrt{\varepsilon}). \tag{4.57}$$

Proof. The second estimate follows right away: because the second relations in (4.55) and (4.56) imply together

$$\|s_4\|_{H^1(b,c)} = \mathcal{O}(\sqrt{\varepsilon}),$$

the compact embedding $H^1(b, c) \subset C[b, c]$ leads to the desired estimate for s_4.

Now we show the first estimate in (4.57). Here, in particular, $s_4(b) = \mathcal{O}(\sqrt{\varepsilon})$. The transmission condition of continuity in (3.28) implies

$$r_4(b) = \mathcal{O}(\sqrt{\varepsilon}). \tag{4.58}$$

Let now $x \in [a, b]$ be arbitrarily fixed. For $y \in [a, x]$, equation (3.26), integrated over $[a, y]$, leads to

$$- \varepsilon r_4'(y) + \varepsilon r_4'(a) + \int_a^y \alpha(t) r_4'(t)\,\mathrm{d}t + \int_a^y \beta(t) r_4(t)\,\mathrm{d}t$$

$$= - \int_a^y \beta(t)(r_3(t) + \ell_\varepsilon(t))\,\mathrm{d}t. \tag{4.59}$$

We multiply this equation by $r_4'(y)$ and integrate over $[a, x]$ to obtain

$$- \varepsilon \int_a^x (r_4'(y))^2\,\mathrm{d}y + \varepsilon r_4'(a) r_4(x) + \int_a^x r_4'(y) \left[\int_a^y \alpha(t) r_4'(t)\,\mathrm{d}t \right] \mathrm{d}y$$

$$+ \int_a^x r_4'(y) \left[\int_a^y \beta(t) r_4(t)\,\mathrm{d}t \right] \mathrm{d}y = - \int_a^x r_4'(y) \left[\int_a^y \beta(t)(r_3(t) + \ell_\varepsilon(t))\,\mathrm{d}t \right] \mathrm{d}y. \tag{4.60}$$

Using integration by parts, we reformulate equation (4.60) as

$$\tilde{a} r_4^2(x) + \tilde{b} r_4(x) + \tilde{c} = 0, \tag{4.61}$$

with coefficients $\tilde{a} = \tilde{a}(x)$, $\tilde{b} = \tilde{b}(r_3, r_4, \ell_\varepsilon, x, \alpha, \beta; \varepsilon)$, $\tilde{c} = \tilde{c}(r_3, r_4, \ell_\varepsilon, x, \alpha, \beta; \varepsilon)$ given by

$$\tilde{a}(x) := \frac{\alpha(x)}{2},$$

$$\tilde{b}(\ldots; \varepsilon) := \varepsilon r_4'(a) - \int_a^x \alpha'(y) r_4(y)\,\mathrm{d}y + \int_a^x \beta(y) r_4(y)\,\mathrm{d}t + \int_a^x [\beta(r_3 + \ell_\varepsilon)](y)\,\mathrm{d}y,$$

$$\tilde{c}(\ldots; \varepsilon) := - \varepsilon \int_a^x (r_4'(y))^2\,\mathrm{d}y - \int_a^x \left[\beta(y) - \frac{\alpha'(y)}{2} \right] r_4^2(y)\,\mathrm{d}y$$

$$- \int_a^x [\beta(r_3 + \ell_\varepsilon) r_4](y)\,\mathrm{d}y.$$

The next step is to prove the estimates

$$|\tilde{b}(\ldots, \varepsilon)|_{C[a,b]} = \mathcal{O}(\sqrt{\varepsilon}) \quad \text{and} \quad |\tilde{c}(\ldots, \varepsilon)|_{C[a,b]} = \mathcal{O}(\varepsilon).$$

Our first attempt is to estimate the term $\varepsilon r_4'(a)$, appearing in the definition of $\tilde{b}$. Integrating (4.59) over $[a, b]$, we get with $r_4(a) = 0$:

$$- \varepsilon r_4'(a)(b - a) = - \varepsilon r_4(b) + \int_a^b \int_a^y [\alpha r_4'](t)\,\mathrm{d}t\,\mathrm{d}y + \int_a^b \int_a^y [\beta r_4](t)\,\mathrm{d}t\,\mathrm{d}y$$

$$+ \int_a^b \int_a^y [\beta(r_3 + \ell_\varepsilon)](t)\,\mathrm{d}t\,\mathrm{d}y.$$

Now we estimate the terms on the right-hand side separately.

First, the relation (4.58) implies $- \varepsilon r_4(b) = \mathcal{O}(\varepsilon\sqrt{\varepsilon})$.

Then, integrating by parts, we get with a positive constant c_1

$$\left|\int_a^b \int_a^y [\alpha r_4'](t)\,\mathrm{d}t\,\mathrm{d}y\right| \leqslant \|\alpha\|_{L^2(a,b)}\|r_4\|_{L^2(a,b)} + (b-a)\|\alpha'\|_{L^2(a,b)}\|r_4\|_{L^2(a,b)} \leqslant c_1\|r_4\|_{L^2(a,b)},$$

consequently this integral is of $\mathcal{O}(\sqrt{\varepsilon})$. Further, there follows

$$\left|\int_a^b \int_a^y \beta(t) r_4(t)\,\mathrm{d}t\,\mathrm{d}y\right| \leqslant \frac{2}{3}(b-a)^{3/2}\|\beta\|_{L^\infty(a,b)}\|r_4\|_{L^2(a,b)} \leqslant c_2\sqrt{\varepsilon},$$

and finally, with a positive constant c_3,

$$\left|\int_a^b \int_a^y [\beta(r_3+\ell_\varepsilon)](t)\,\mathrm{d}t\,\mathrm{d}y\right| \leqslant \frac{2}{3}(b-a)^{3/2}\|\beta\|_{L^\infty(a,b)}(\|r_3\|_{L^2(a,b)} + \|\ell_\varepsilon\|_{L^2(a,b)}) \leqslant c_3\sqrt{\varepsilon},$$

hence the last integral is also of order $\mathcal{O}(\sqrt{\varepsilon})$. Collecting these estimates, we obtain

$$-\varepsilon r_4'(a) = \mathcal{O}(\sqrt{\varepsilon}).$$

Let us now return to the evaluation of the coefficients $\tilde{b}$ and $\tilde{c}$ on $[a,b]$. We point out that all the norms appearing here are considered over $[a,b]$. We use the last relation, the Schwarz inequality, as well as the estimates (3.21), (4.51) and (4.55). Then

$$|\tilde{b}(\ldots,\varepsilon)| \leqslant \mathcal{O}(\sqrt{\varepsilon}) + \underbrace{\|\alpha'\|_{L^2}}_{=\mathcal{O}(1)}\cdot\underbrace{\|r_4\|_{L^2}}_{=\mathcal{O}(\sqrt{\varepsilon})} + \|\beta\|_{L^2}[\underbrace{\|r_4\|_{L^2}}_{=\mathcal{O}(\sqrt{\varepsilon})} + \underbrace{\|r_3\|_{L^2}}_{=\mathcal{O}(\varepsilon)} + \underbrace{\|\ell_\varepsilon\|_{L^2}}_{=\mathcal{O}(\sqrt{\varepsilon})}] = \mathcal{O}(\sqrt{\varepsilon}).$$

Using now the first estimate in (4.56), as well as (3.21), (4.51) and (4.55), we obtain

$$|\tilde{c}(\ldots,\varepsilon)| \leqslant \varepsilon\underbrace{\|r_4'\|_{L^2}^2}_{=\mathcal{O}(1)} + \underbrace{\left\|\beta-\frac{\alpha'}{2}\right\|_{L^2}}_{=\mathcal{O}(1)}\cdot\underbrace{\|r_4\|_{L^2}^2}_{=\mathcal{O}(\varepsilon)} + \|\beta\|_{L^\infty}\underbrace{\|r_4\|_{L^2}}_{=\mathcal{O}(\sqrt{\varepsilon})}\cdot(\underbrace{\|r_3\|_{L^2}}_{=\mathcal{O}(\varepsilon)} + \underbrace{\|\ell_\varepsilon\|_{L^2}}_{=\mathcal{O}(\sqrt{\varepsilon})}) = \mathcal{O}(\varepsilon).$$

In consequence, equation (4.61) takes the asymptotic form

$$r_4^2(x) + \mathcal{O}(\sqrt{\varepsilon})\,r_4(x) + \mathcal{O}(\varepsilon) = 0 \quad \text{for} \quad x \in [a,b],$$

since $\alpha(x)/2 \geqslant \alpha_0/2 > 0$ on $[a,b]$. Hence, there exists a positive constant C, such that

$$|r_4(x)| \leqslant C\sqrt{\varepsilon} \quad \text{for all} \quad x \in [a,b],$$

the first estimate in (4.57) is also proved. This completes the proof of the theorem. □

We summarize the results obtained in the case $\beta \geqslant 0$, $\beta - \frac{1}{2}\alpha' \geqslant 0$ in the following.

Theorem 4.7. *Assume that the coefficients* α *and* β *satisfy* (4.53). *Let* $f \in L^\infty(a, b)$ *and* f *be continuous at* b; *let* $g \in L^2(b, c)$. *Further let* $(u_\varepsilon, v_\varepsilon) \in H^2(a, b) \times H^2(b, c)$ *be the solution of the elliptic–elliptic boundary value problem with* $\varepsilon > 0$, *and let* $(u_0, v_0) \in H^1(a, b) \times H^2(b, c)$ *be the solution of the degenerate hyperbolic–elliptic boundary value problem. Then, as* $\varepsilon \to 0$, *the remainders* $(r_\varepsilon, s_\varepsilon)$ *satisfy the relations*

$$|r_\varepsilon|_{C[a,b]} = |u_\varepsilon - u_0 - \ell_\varepsilon|_{C[a,b]} = \mathcal{O}(\sqrt{\varepsilon}), \qquad |s_\varepsilon|_{C[b,c]} = |v_\varepsilon - v_0|_{C[b,c]} = \mathcal{O}(\sqrt{\varepsilon}). \tag{4.62}$$

Proof. The proof follows directly from Lemma 4.5 and from the estimates regarding (R_4). The relations (4.62) are immediate consequences of the representations in (3.24) and of the estimates (4.49), (4.57), respectively. □

5. Numerical results

For numerical computations, we approximate the solution of the coupled elliptic–elliptic boundary value problem by solving the degenerate hyperbolic–elliptic problem and then updating the corresponding solution in an appropriate way by boundary layer terms. We recall that the method gives first informations on the *a posteriori* correction of the solution of a Navier–Stokes/Euler coupling by using boundary layer terms.

Numerical tests have been performed for both cases $\beta \equiv 0$, $\alpha' \leqslant 0$ and $\beta \geqslant 0$, $\beta - \frac{1}{2}\alpha' \geqslant 0$. We mention that also in the second case, the use of the boundary layer function ℓ_ε, defined for $\beta \equiv 0$, agrees very well with the asymptotic convergence results.

The hyperbolic component of the degenerate problem (P_0) can be solved explicitly:

$$u_0(x) = \int_a^x \frac{f(t)}{\alpha(t)} \exp\left[- \int_t^x \frac{\beta(\tau)}{\alpha(\tau)} \, d\tau \right] dt \quad \text{for} \quad x \in [a, b].$$

Then $u_{0,h}$ is computed by using the six-point Gaussian quadrature.

With $\gamma(u_0) := \alpha(b) u_0(b)$, the elliptic boundary value problem on $[b, c]$ is given by equation (2.5), together with the boundary conditions $-\mu v_0'(b) + \alpha(b) v_0(b) = \gamma(u_0)$ and $v_0(c) = 0$. Here we use a standard Galerkin–Bubnov finite element method with piece-wise linear trial and test functions vanishing at $x = c$, applied to

Find $v_0 \in V := \{v \in H^1(b, c) \colon v(c) = 0\}$ *such that*

$$\mu \int_b^c v_0' \psi' \, dx - \int_b^c \alpha v_0 \psi' \, dx + \int_b^c [\beta - \alpha'] v_0 \psi \, dx$$
$$= \int_b^c g \psi \, dx + \gamma(u_0) \psi(b) \quad \textit{for all} \quad \psi \in V.$$

We define now the *discrete asymptotic solution* $(u_{\varepsilon,h}, v_{\varepsilon,h})$ by

$$(u_{\varepsilon,h}, v_{\varepsilon,h}) := (u_{0,h}, v_{0,h}) + (\ell_{\varepsilon,h}, 0) \quad \text{in} \quad [a,b]\times[b,c]. \tag{5.1}$$

The function $\ell_{\varepsilon,h}$ represents a discrete approximation of the boundary layer function in (3.8) and is also calculated by means of the six-point Gaussian quadrature.

The discrete asymptotic solution should be an approximation of the solution to the full elliptic–elliptic coupled boundary value problem. In the one-dimensional case, we may examine the accuracy of this solution by using the Galerkin–Bubnov finite element method for the elliptic–elliptic coupled problem on $[a,c]$ (with piecewise linear trial and test functions vanishing at $x=a$ and at $x=c$). Let $(\tilde{u}_{\varepsilon,h}, \tilde{v}_{\varepsilon,h})$ denote the global FEM-solution.

Figures 1–3 show the approximate solution $(u_{0,h}, v_{0,h})$ of the degenerate problem, together with the numerical solutions of the elliptic–elliptic coupled problem, obtained by

- updating $(u_{0,h}, v_{0,h})$ with boundary layer terms as in (5.1);
- direct FEM-discretization of the coupled problem on $[a,c]$.

In our tests we choose $a=-1$, $b=0$, $c=1$. The viscosity coefficients are $\varepsilon_1 = 0.0005$, $\varepsilon_2 = 0.002$, $\varepsilon_3 = 0.006$, the coefficient function α is given by $\alpha(x) := (x-1.5)^2$; and the right-hand side is given by $f(x) = -(x+0.4)\cos(10(x+0.75))$, $x\in[-1,1]$. The numerical solutions to the cases $\beta\equiv 0$ and $\beta(x) := (x+6)^2 \geqslant 0$ are presented in Figs. 1 and 2, respectively. Note that the coefficients α and β satisfy in both cases the property (4.53).

In these figures, the solution of the degenerate problem is plotted as a solid line. The updated solutions, corresponding to different values of ε, are compared with the FEM-solution obtained directly on $[a,c]$. Since the discrete remainders $(r_{\varepsilon,h}, s_{\varepsilon,h})$ are contained in the FEM-solution $(\tilde{u}_{\varepsilon,h}, \tilde{v}_{\varepsilon,h})$ and neglected in the definition of the discrete asymptotic solution $(u_{\varepsilon,h}, v_{\varepsilon,h})$, there follows that the function

$$\eta_{\varepsilon,h}(x) := (\tilde{u}_{\varepsilon,h}, \tilde{v}_{\varepsilon,h})(x) - (u_{\varepsilon,h}, v_{\varepsilon,h})(x) \quad \text{for} \quad x\in[a,c]$$

expresses the behaviour of the remainders on $[a,c]$ at the discrete level. Of course, this behaviour depends on ε, as can be seen in Fig. 3 (above).

We emphasize that the transmission condition in (2.6),

$$\alpha(b)u_0(b) = -\mu v_0'(b) + \alpha(b)v_0(b) \quad \text{at} \quad x=b,$$

allows a discontinuity $v_0(b) - u_0(b)$ (depending on the viscosity coefficient μ), which is plotted in Fig. 3 (below) for different coefficients $\mu\in[0.01, 2]$.

Remark on the "*Vanishing viscosity method*" for solving the degenerate problem. Since we can discretize also the elliptic–elliptic coupled boundary value problem, we are able to analyse the connection between these two coupled problems *inversely*. Often the solution of the degenerate hyperbolic–elliptic transmission-boundary value problem is approximated by solving numerically regularized problems corresponding to artificial viscosities $0<\varepsilon\to 0$, i.e. a discrete counterpart to the vanishing viscosity method. In this framework, we improve the L^2-convergence result by Gastaldi and Quarteroni [13] by using boundary layer functions: in the case of L^∞-right-hand

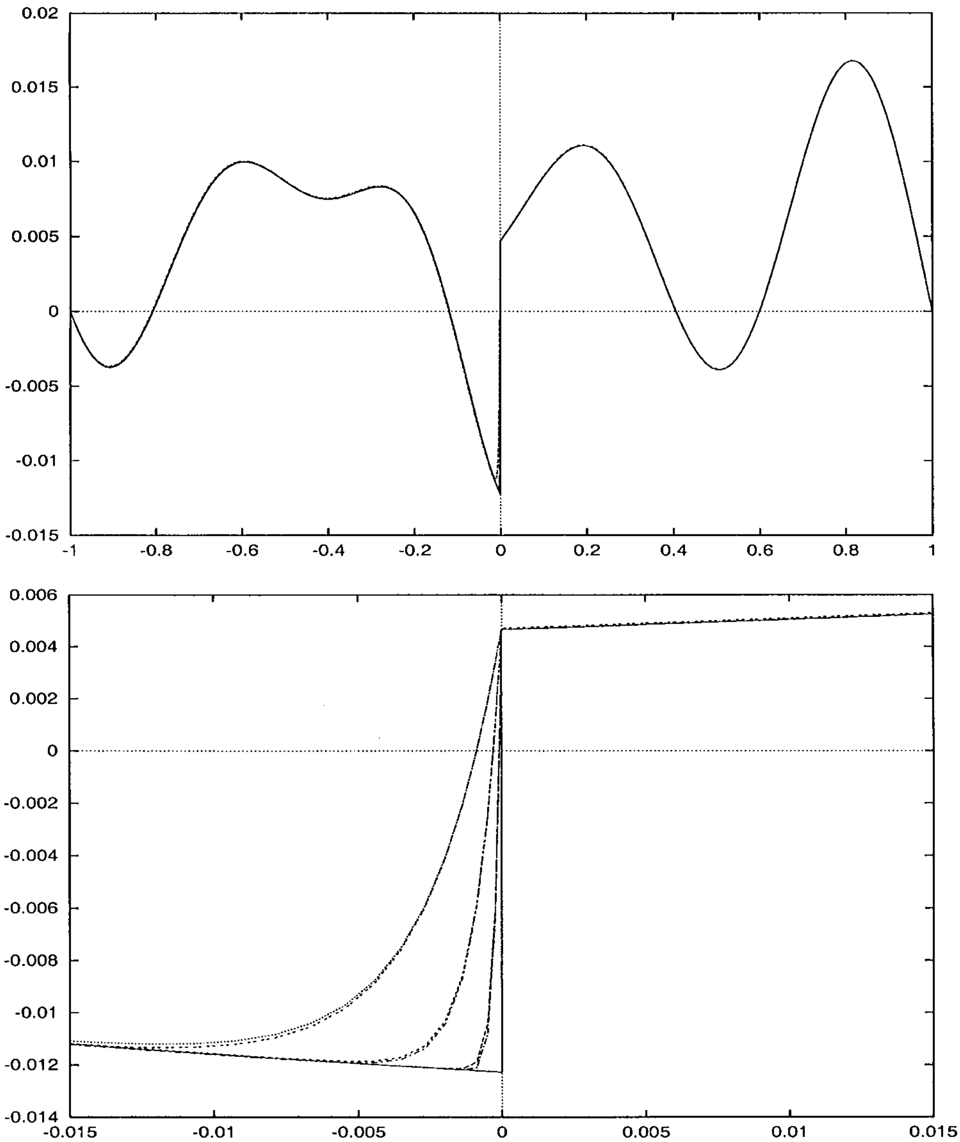

Fig. 1. Approximate solutions in the case $\alpha(x) = (x - 1.1)^2$, $\beta \equiv 0$, $\mu = 1$

sides, which are continuous at the interface point $x = b$, the solution of the degenerate problem is the *uniform* limit of a sequence of functions obtained by subtracting the boundary layer terms ℓ_ε from the solutions to regularized problems (P_ε).

Note that the jump $v_0(b) - u_0(b)$ is included in the expression of the boundary layer function (3.8). This jump is unknown, because the solution of the hyperbolic–elliptic problem is unknown (this is what we want to determine here). However, we can

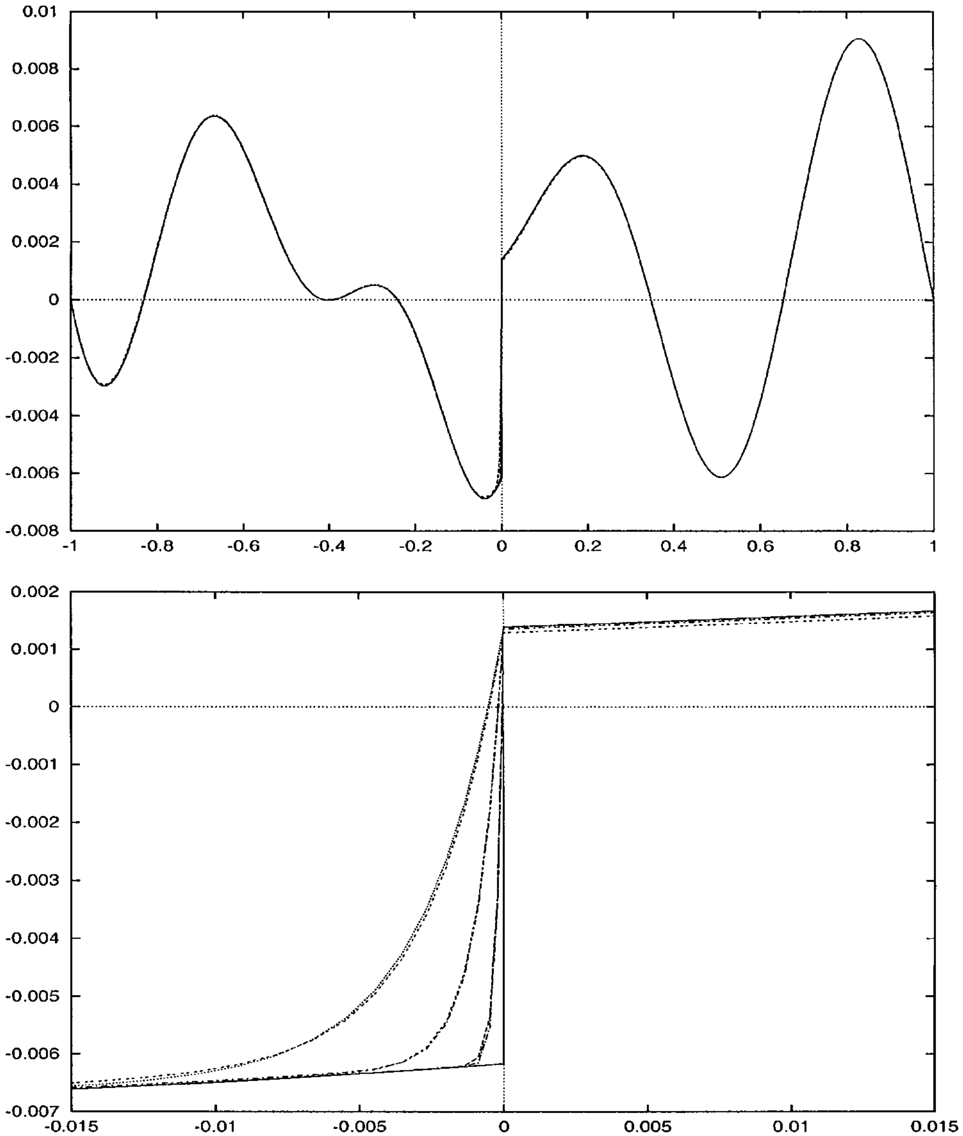

Fig. 2. Approximate solutions in the case $\alpha(x) = (x - 1.1)^2$, $\beta(x) = (x + 5)^2$, $\mu = 1$

overcome this difficulty by computing this jump *asymptotically* by using approximate solutions of regularized problems with different viscosity coefficients. Let us denote by $(u_{\varepsilon_1}, v_{\varepsilon_1})$ and $(u_{\varepsilon_2}, v_{\varepsilon_2})$ the unique solutions to $(\mathrm{P}_{\varepsilon_1})$ and $(\mathrm{P}_{\varepsilon_2})$ for given $0 < \varepsilon_1 < \varepsilon_2$. Due to (3.1), the solution components u_{ε_1} and u_{ε_2} admit the representations

$$u_{\varepsilon_1}(x) = u_0(x) + \ell_{\varepsilon_1}(x) + r_{\varepsilon_1}(x), \qquad u_{\varepsilon_2}(x) = u_0(x) + \ell_{\varepsilon_2}(x) + r_{\varepsilon_2}(x)$$
$$\text{for} \quad x \in [a, b],$$

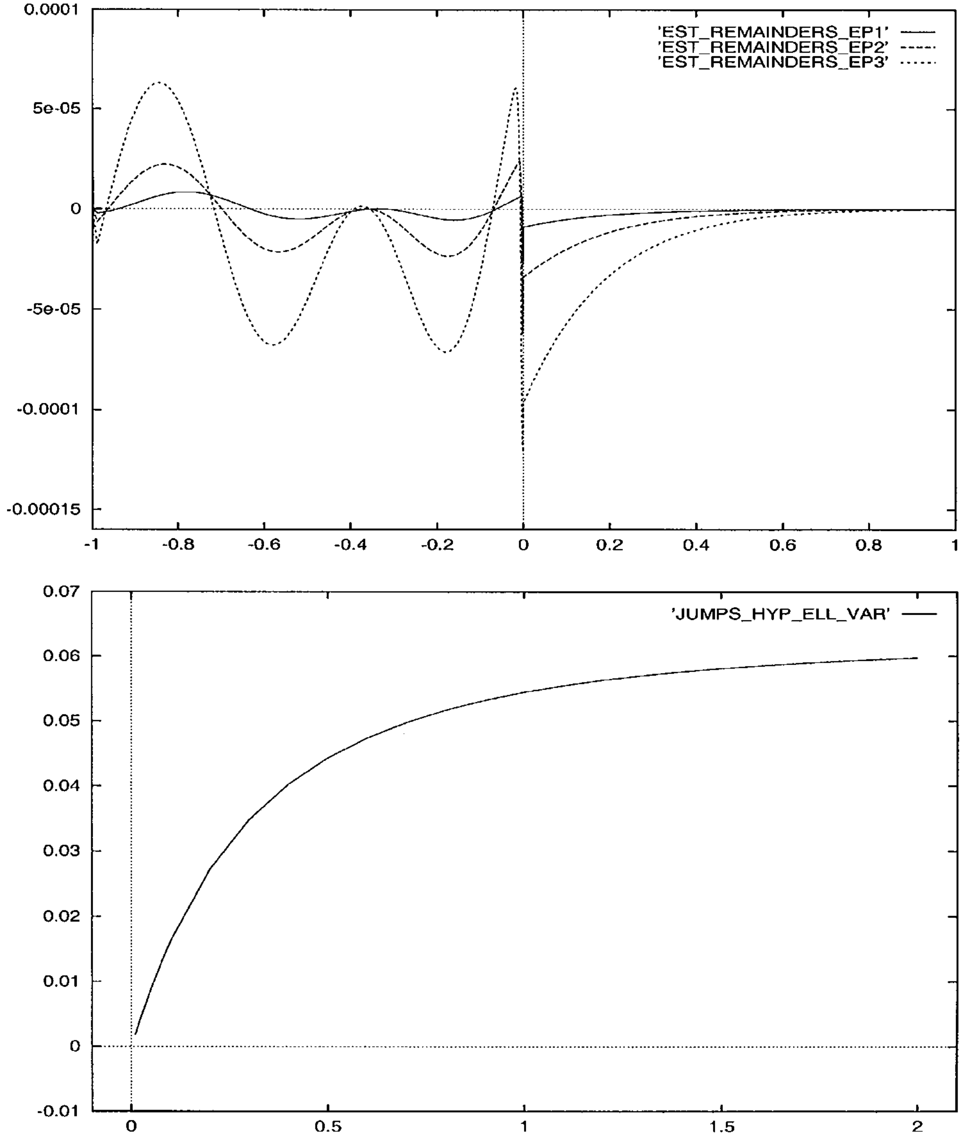

Fig. 3. Discrete remainders $\eta_{\varepsilon,h}(x)$ (above); the jump dependence on μ (below)

where u_0 denotes the (same) solution component for the hyperbolic–elliptic coupled boundary value problem, the boundary layer functions are defined as in (3.8), and, finally, the remainders $r_{\varepsilon_1}(x)$ and $r_{\varepsilon_2}(x)$ satisfy the estimates (4.31), (4.62). Subtracting these representations we get

$$u_{\varepsilon_1}(x) - u_{\varepsilon_2}(x) = C\alpha(b) \cdot h_{\varepsilon_1,\varepsilon_2}(x) + r_{\varepsilon_1}(x) - r_{\varepsilon_2}(x) \quad \text{for} \quad x \in [a, b],$$

where the function $h_{\varepsilon_1,\varepsilon_2}(x)$ is defined by

$$h_{\varepsilon_1,\varepsilon_2}(x) := \int_0^{(b-x)/\varepsilon_2} \exp\left[-\int_0^s \alpha(b-\varepsilon_2 u)\,\mathrm{d}u \right] \mathrm{d}s - \int_0^{(b-x)/\varepsilon_1} \exp\left[-\int_0^s \alpha(b-\varepsilon_1 u)\,\mathrm{d}u \right] \mathrm{d}s \quad \text{for} \quad x \in [a,b].$$

Since the remainders satisfy the estimate $|r_{\varepsilon_1} - r_{\varepsilon_2}|_{C[a,b]} = \mathcal{O}(\sqrt{\varepsilon_2})$, we define the *asymptotic approximation of the jump C* by

$$C = C(\varepsilon_1, \varepsilon_2, \mu) := \frac{u_{\varepsilon_1}(x^*) - u_{\varepsilon_2}(x^*)}{\alpha(b)\, h_{\varepsilon_1,\varepsilon_2}(x^*)}, \tag{5.2}$$

where $x^* \in (a,b)$ denotes the point where $|h_{\varepsilon_1,\varepsilon_2}|$ becomes maximal. This point is obtained via the equivalence

$$h'_{\varepsilon_1,\varepsilon_2}(x^*) = 0 \Leftrightarrow \int_{t=x^*}^{b} \alpha(t)\,\mathrm{d}t = \frac{\varepsilon_1\varepsilon_2}{\varepsilon_2 - \varepsilon_1} \log\frac{\varepsilon_2}{\varepsilon_1}. \tag{5.3}$$

Because of $\alpha(t) \geqslant \alpha_0 > 0$, the point x^* is uniquely defined by (5.3).
We now define the approximate solution (u_0, v_0) by subtracting from u_{ε_1} the boundary layer function ℓ_{ε_1} multiplied by the corresponding approximate jump (5.2).

Figure 4 presents a comparison between the jumps obtained at $x = b$ by solving the hyperbolic–elliptic coupled problem directly (solid line) and the 'asymptotic' jumps (5.2) (dashed line). We observe a very good agreement between the two curves showing the dependence of the jumps with respect to $\mu \in [0.01, 2]$. Finally, the numerical evaluation $(u_{0,h}, v_{0,h})$ of the exact solution (u_0, v_0) to the reduced problem is compared with the (uniform) limit of solutions to the regularized problem by using the asymptotic jumps.

6. Concluding remarks

Here we perform singular perturbation analysis for a coupled elliptic–elliptic *transmission*-boundary value problem. We use as a first approximation the solution of the corresponding hyperbolic–elliptic degenerate transmission problem, obtained by dropping the viscosity terms in one of the subregions. This solution is corrected by appropriate boundary layer functions, such that finally the natural transmission conditions are fulfilled asymptotically. We show the validity of the asymptotic expansions, i.e. the remainders converge to zero uniformly. Some numerical tests are presented. The paper can be considered as an intermediate step in the analysis of coupled models in Computational Fluid Dynamics.

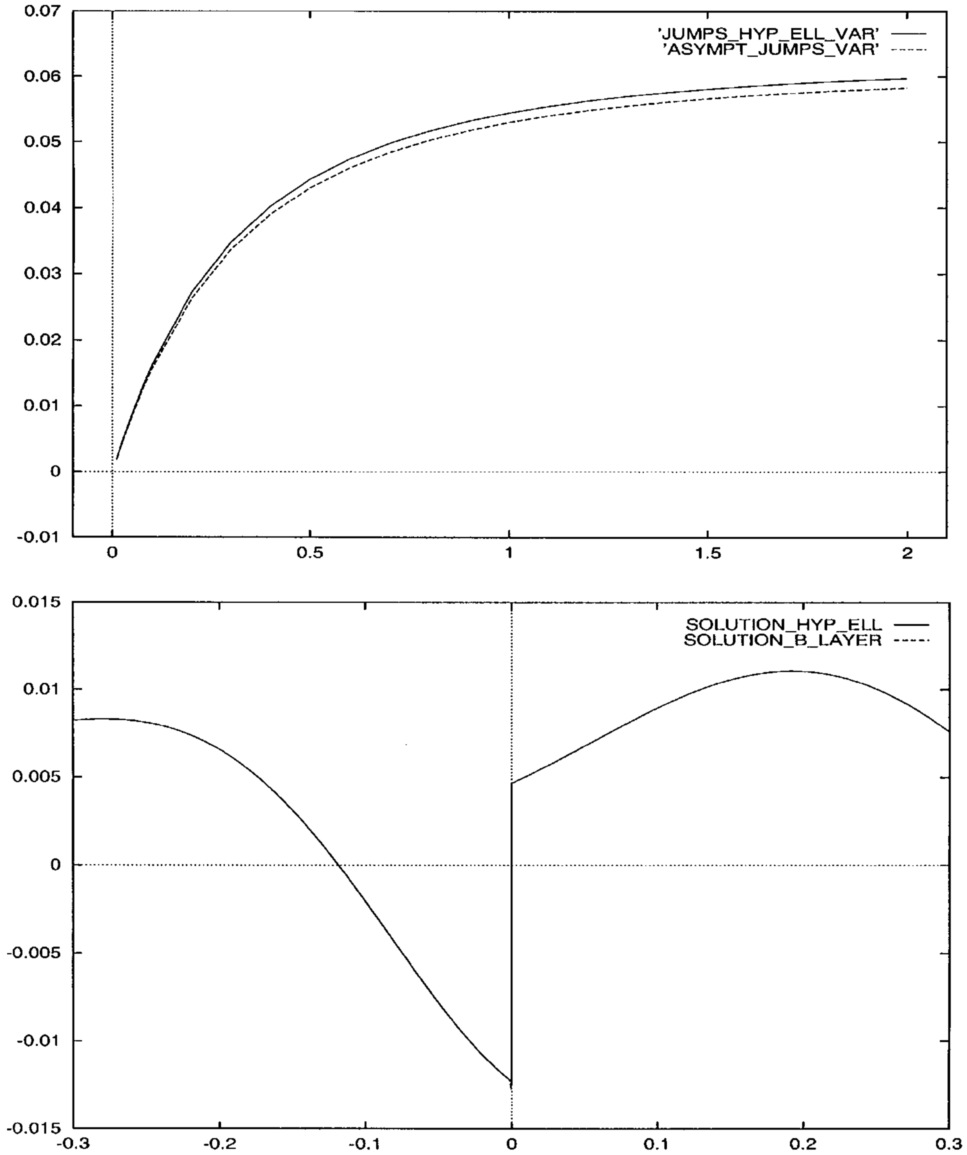

Fig. 4. Comparison between the ‘exact’ and the ‘asymptotic’ jumps (above); comparison between the ‘exact’ and the ‘asymptotic’ solution of the degenerate problem (below)

Appendix A

Proof of Lemma 3.1. With $C := v_0(b) - u_0(b)$ (note that C depends on f), one obtains

$$|\ell_\varepsilon(a)| = |C| \cdot \left| 1 - \alpha(b) \int_0^{(b-a)/\varepsilon} \mathrm{e}^{-\int_0^s \alpha(b - \varepsilon u)\,\mathrm{d}u}\, \mathrm{d}s \right|$$

$$= |C|\cdot\left|\alpha(b)\int_0^{\infty} e^{-s\alpha(b)}\,ds - \alpha(b)\int_0^{(b-a)/\varepsilon} e^{-\int_0^s \alpha(b-\varepsilon u)\,du}\,ds\right|$$

$$= |C|\,\alpha(b)\left|\int_0^{(b-a)/\varepsilon} e^{-s\alpha(b)}\,ds + \int_{(b-a)/\varepsilon}^{\infty} e^{-s\alpha(b)}\,ds - \int_0^{(b-a)/\varepsilon} e^{-\int_0^s \alpha(b-\varepsilon u)\,du}\,ds\right|$$

$$\leqslant C\alpha(b)\Bigg[\underbrace{\left|\int_0^{(b-a)/\varepsilon}\left[e^{-s\alpha(b)} - e^{-\int_0^s \alpha(b-\varepsilon u)\,du}\right]ds\right|}_{=:I_1(\varepsilon)} + \underbrace{\left|\int_{(b-a)/\varepsilon}^{\infty} e^{-s\alpha(b)}\,ds\right|}_{=:I_2(\varepsilon)}\Bigg].$$

We first write $I_1(\varepsilon)$ in the equivalent form

$$I_1(\varepsilon) = \int_0^{(b-a)/\varepsilon} e^{-s\alpha(b)}\left[1 - e^{\int_0^s \alpha(b)\,du - \int_0^s \alpha(b-\varepsilon u)\,du}\right] ds.$$

Using the Lipschitz continuity of α, one gets for $\eta(\varepsilon, s) := \int_0^s(\alpha(b) - \alpha(b-\varepsilon u))\,du$ the estimate $|\eta(\varepsilon, s)| \leqslant \int_0^s |\alpha(b) - \alpha(b-\varepsilon u)|\,du \leqslant \int_0^s L\varepsilon u\,du = Ls^2/2\cdot\varepsilon \leqslant 1$ for sufficiently small $\varepsilon > 0$. Consequently, $1 - e^{\eta(\varepsilon,s)} = -\eta(\varepsilon, s) - \frac{1}{2}\eta^2(\varepsilon, s) - \cdots$, hence,

$$|1 - e^{\eta(\varepsilon,s)}| \leqslant |\eta(\varepsilon, s)| + \mathcal{O}(|\eta(\varepsilon, s)|^2) = \frac{Ls^2}{2}\varepsilon + s^4\,\mathcal{O}(\varepsilon^2).$$

We obtain

$$|I_1(\varepsilon)| \leqslant \int_0^{(b-a)/\varepsilon} e^{-s\alpha(b)}\cdot|1 - e^{\eta(\varepsilon,s)}|\,ds$$

$$\leqslant \int_0^{(b-a)/\varepsilon} e^{-s\alpha(b)}\left[\frac{L\varepsilon}{2}s^2 + \mathcal{O}(\varepsilon^2)s^4\right]ds = \mathcal{O}(\varepsilon).$$

For the second integral we get $I_2(\varepsilon) = (1/\alpha(b))\,e^{-\alpha(b)\cdot(b-a)/\varepsilon} = \mathcal{O}(e^{-\alpha(b)\cdot(b-a)/\varepsilon})$. The last relations, together with the representation of $\ell_\varepsilon(a)$ lead to

$$\ell_\varepsilon(a) = \mathcal{O}(\varepsilon),$$

where the constant in $\mathcal{O}(\varepsilon)$ depends on f. This completes the proof. □

Appendix B

Proof of Lemma 3.2. We begin with a decomposition of ℓ_ε which will allow the desired estimate. With $C := v_0(b) - u_0(b)$, we have

$$\ell_\varepsilon(x) = \ell_\varepsilon^{(1)}(x) + \ell_\varepsilon^{(2)}(x), \quad x \in [a, b], \tag{B.1}$$

where $\ell_\varepsilon^{(1)}$, $\ell_\varepsilon^{(2)}$ are given by

$$\ell_\varepsilon^{(1)}(x) := C\alpha(b)\int_0^{(b-x)/\varepsilon}\left(e^{-\int_0^s \alpha(b)\,du} - e^{-\int_0^s \alpha(b-\varepsilon u)\,du}\right) ds,$$

$$\ell_\varepsilon^{(2)}(x) := C\alpha(b)\int_{(b-x)/\varepsilon}^{\infty} \mathrm{e}^{-\int_0^s \alpha(b)\,\mathrm{d}u}\,\mathrm{d}s.$$

Using similar arguments as in Appendix A, we get

$$|\ell_\varepsilon^{(1)}(x)| \leqslant |C|\cdot\alpha(b)\cdot\int_0^{(b-a)/\varepsilon} \mathrm{e}^{-s\alpha(b)}\cdot\left[\frac{Ls^2}{2}\varepsilon + s^4\cdot\mathcal{O}(\varepsilon^2)\right]\mathrm{d}s \quad \text{for all } x\in[a,b].$$

Integrating by parts, we find with a positive constant K the relation $|\ell_\varepsilon^{(1)}(x)| \leqslant K\varepsilon$ for all $x\in[a,b]$, hence

$$\|\ell_\varepsilon^{(1)}\|_{L^2(a,b)} = \mathcal{O}(\varepsilon). \tag{B.2}$$

Since $\ell_\varepsilon^{(2)}(x) = C\exp[-\alpha(b)(b-x)/\varepsilon]$, we find

$$\|\ell_\varepsilon^{(2)}\|_{L^2(a,b)} = \mathcal{O}(\sqrt{\varepsilon}). \tag{B.3}$$

With (B.2) and (B.3), the proof of Lemma 3.2 is completed. □

Appendix C

We prove here the estimate (4.30) for the function $r_{2,2}$, defined in (4.26) (Fig. 5). We write

$$r_{2,2}(x) = r_{2,3}(x) + r_{2,4}(x) \quad \text{for} \quad x\in[a,b], \tag{C.1}$$

where the functions $r_{2,3}$ and $r_{2,4}$ are defined for $x\in[a,b]$ by

$$r_{2,3}(x) := \int_{t=a}^{x}\left[\int_{\sigma=t}^{x} u_0''(\sigma)\,\mathrm{e}^{-(1/\varepsilon)\int_{\tau=t}^{\sigma}\alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right]\mathrm{d}t, \tag{C.2}$$

$$r_{2,4}(x) := \int_{t=a}^{x}\left[\int_{\sigma=x}^{b} u_0''(\sigma)\,\mathrm{e}^{-(1/\varepsilon)\int_{\tau=t}^{\sigma}\alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right]\mathrm{d}t. \tag{C.3}$$

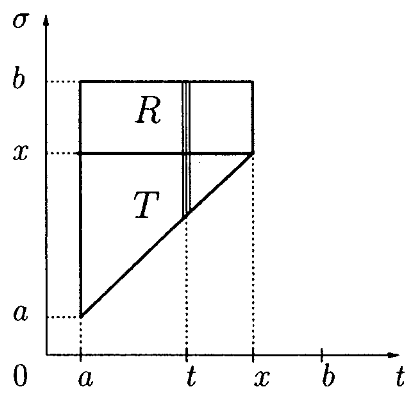

Fig. 5. Domain of integration

Further, we show the estimates

$$
\begin{aligned}
&|r_{2,3}|_{C[a,b]} = \mathcal{O}(\varepsilon), \\
&|r_{2,4}|_{C[a,b]} = \mathcal{O}(\varepsilon).
\end{aligned} \tag{C.4}
$$

C.1. *Estimate for* $r_{2,3}$

Changing in (C.2) the order of integration on the triangle T: $a \leqslant t \leqslant \sigma \leqslant x$ and then integrating by parts, we obtain

$$
\begin{aligned}
r_{2,3}(x) = \int_{\sigma=a}^{x} u_0''(\sigma)\left[\int_{t=a}^{\sigma} \mathrm{e}^{-(1/\varepsilon)\int_{\tau=t}^{\sigma}\alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t\right]\mathrm{d}\sigma &= \underbrace{u_0'(x)\int_{t=a}^{x} \mathrm{e}^{-(1/\varepsilon)\int_{\tau=t}^{x}\alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t}_{=:I_1(x,\varepsilon)} \\
&+ \underbrace{\int_a^x u_0'(\sigma)\int_a^{\sigma}\frac{\alpha(\sigma)-\alpha(t)}{\varepsilon}\,\mathrm{e}^{-(1/\varepsilon)\int_t^{\sigma}\alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t\,\mathrm{d}\sigma}_{=:I_2(x,\varepsilon)} \\
&+ \underbrace{\int_0^x u_0'(\sigma)\left[\int_a^{\sigma}\frac{\alpha(t)}{\varepsilon}\mathrm{e}^{-(1/\varepsilon)\int_t^{\sigma}\alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t - 1\right]\mathrm{d}\sigma.}_{=:I_3(x,\varepsilon)}
\end{aligned} \tag{C.5}
$$

We show now that $I_j(x, \varepsilon)$ are of $\mathcal{O}(\varepsilon)$, where the constants in $\mathcal{O}(\varepsilon)$ depend on M.

For every $x \in [a, b]$ there follows

$$
|I_1(x,\varepsilon)| \leqslant |u_0'(x)| \int_{t=a}^{x} \mathrm{e}^{-(\alpha_0/\varepsilon)(x-t)}\,\mathrm{d}t \leqslant \frac{M}{\alpha_0^2}\,\varepsilon.
$$

Furthermore,

$$
\begin{aligned}
|I_2(x,\varepsilon)| &\leqslant \int_a^x |u_0'(\sigma)| \int_a^{\sigma} \frac{|\alpha(\sigma)-\alpha(t)|}{\varepsilon}\,\mathrm{e}^{-(\alpha_0/\varepsilon)(\sigma-t)}\,\mathrm{d}t\,\mathrm{d}\sigma \\
&\leqslant \frac{ML}{\alpha_0\varepsilon}\int_a^x \mathrm{e}^{-\alpha_0\sigma/\varepsilon}\int_a^{\sigma}(\sigma-t)\,\mathrm{e}^{-\alpha_0 t/\varepsilon}\,\mathrm{d}t\,\mathrm{d}\sigma \leqslant \Xi_1(x,\varepsilon)\,\varepsilon,
\end{aligned}
$$

where the expression $\Xi_1(x,\varepsilon) := (ML/\alpha_0^3)(x-a) - (2ML/\alpha_0^4)\varepsilon + (ML(x-a)/\alpha_0^3)\,\mathrm{e}^{-\alpha_0(x-a)/\varepsilon} + (2ML/\alpha_0^4)\mathrm{e}^{-\alpha_0(x-a)/\varepsilon}\varepsilon$ is uniformly bounded for $\varepsilon \to 0$.

In order to estimate $I_3(x,\varepsilon)$, we first note that $\int_{t=a}^{\sigma}(\alpha(t)/\varepsilon)\mathrm{e}^{-(1/\varepsilon)\int_{\tau=t}^{\sigma}\alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}t - 1 = -\,\mathrm{e}^{-(1/\varepsilon)\int_{\tau=a}^{\sigma}\alpha(\tau)\,\mathrm{d}\tau}$, and, consequently,

$$
|I_3(x,\varepsilon)| \leqslant \int_a^x |u_0'(\sigma)|\cdot \mathrm{e}^{-(1/\varepsilon)\int_a^{\sigma}\alpha(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma \leqslant \frac{M}{\alpha_0}\int_a^x \mathrm{e}^{-(\alpha_0/\varepsilon)(\sigma-a)}\,\mathrm{d}\sigma = \Xi_2(x,\varepsilon)\,\varepsilon, \tag{C.6}
$$

where $\Xi_2(x,\varepsilon) := M(1-\mathrm{e}^{-\alpha_0(x-a)/\varepsilon})/\alpha_0^2$ is uniformly bounded for $\varepsilon \to 0$. The representation (C.5), together with the above estimates, imply the first estimate in (C.4).

C.2. *Estimate for* $r_{2,4}$

Using Fubini's theorem in the rectangle $R := [a, x] \times [x, b]$ and integrating by parts, we find

$$r_{2,4}(x) = \int_{\sigma=x}^{b} u_0''(\sigma) \left[\int_{t=a}^{x} e^{-(1/\varepsilon)\int_{\tau=t}^{b} \alpha(\tau)\,d\tau}\, dt \right] d\sigma = I_4(x, \varepsilon) + I_5(x, \varepsilon) + I_6(x, \varepsilon), \tag{C.7}$$

where $I_4(x, \varepsilon)$, $I_5(x, \varepsilon)$ and $I_6(x, \varepsilon)$ are given by

$$I_4(x, \varepsilon) := u_0'(b) \int_a^x e^{-(1/\varepsilon)\int_{\tau=t}^{b} \alpha(\tau)\,d\tau}\, dt - u_0'(x) \int_a^x e^{-(1/\varepsilon)\int_{\tau=t}^{x} \alpha(\tau)\,d\tau}\, dt,$$

$$I_5(x, \varepsilon) := \int_{\sigma=x}^{b} u_0'(\sigma) \left[\int_{t=a}^{x} \frac{\alpha(\sigma) - \alpha(t)}{\varepsilon} e^{-(1/\varepsilon)\int_{\tau=t}^{\sigma} \alpha(\tau)\,d\tau}\, dt \right] d\sigma,$$

$$I_6(x, \varepsilon) := \int_{\sigma=x}^{b} u_0'(\sigma) \left[\int_{t=a}^{x} \frac{\alpha(t)}{\varepsilon} e^{-(1/\varepsilon)\int_{\tau=t}^{\sigma} \alpha(\tau)\,d\tau}\, dt \right] d\sigma \quad \text{for} \quad x \in [a, b].$$

Using similar arguments we obtain for all $x \in [a, b]$ the inequalities

$$|I_4(x, \varepsilon)| \leqslant \frac{M}{{\alpha_0}^2} \left[(e^{-\alpha_0(b-x)/\varepsilon} - e^{-\alpha_0(b-a)/\varepsilon}) + (1 - e^{-\alpha_0(x-a)/\varepsilon}) \right] \cdot \varepsilon,$$

$$\begin{aligned}
|I_5(x, \varepsilon)| \leqslant{}& \frac{2ML}{\alpha_0^4} \varepsilon^2 - \frac{ML}{\alpha_0^3} \left(b - x + 2\frac{\varepsilon}{\alpha_0} \right) e^{-\alpha_0(b-x)/\varepsilon} \cdot \varepsilon \\
&+ \frac{ML}{\alpha_0^3} \left(b - a + 2\frac{\varepsilon}{\alpha_0} \right) e^{-\alpha_0(b-a)/\varepsilon} \cdot \varepsilon \\
&- \frac{ML}{\alpha_0^3} \left(x - a + 2\frac{\varepsilon}{\alpha_0} \right) e^{-\alpha_0(x-a)/\varepsilon} \cdot \varepsilon.
\end{aligned}$$

Finally,

$$I_6(x, \varepsilon) = \int_x^b u_0'(\sigma) \cdot e^{-(1/\varepsilon)\int_{\tau=x}^{\sigma} \alpha(\tau)\,d\tau}\, d\sigma - \int_x^b u_0'(\sigma)\, e^{-(1/\varepsilon)\int_{\tau=a}^{\sigma} \alpha(\tau)\,d\tau}\, d\sigma,$$

and, consequently,

$$\begin{aligned}
|I_6(x, \varepsilon)| &\leqslant \frac{M}{\alpha_0} \left[\int_x^b e^{-(\alpha_0/\varepsilon)(\sigma - x)}\, d\sigma + \int_{\sigma=x}^{b} e^{-(\alpha_0/\varepsilon)(\sigma - a)}\, d\sigma \right] \\
&= \frac{M}{\alpha_0^2} [1 - e^{-\alpha_0(b-x)/\varepsilon} + e^{-\alpha_0(x-a)/\varepsilon} - e^{-\alpha_0(b-a)/\varepsilon}] \cdot \varepsilon.
\end{aligned}$$

These estimates, together with the representation (C.7) lead to the second estimate in (C.4) for the function $r_{2,4}$. □

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