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Inexact Halpern-type proximal point algorithm

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Abstract We present several strong convergence results for the modified, Halpern-type, proximal point algorithm $x_{n+1} = \alpha_n u + (1 - \alpha_n)J_{\beta_n} x_n + e_n$ ($n = 0, 1, \ldots; u, x_0 \in H$ given, and $J_{\beta_n} = (I + \beta_n A)^{-1}$, for a maximal monotone operator A) in a real Hilbert space, under new sets of conditions on $\alpha_n \in (0, 1)$ and $\beta_n \in (0, \infty)$. These conditions are weaker than those known to us and our results extend and improve some recent results such as those of H. K. Xu. We also show how to apply our results to approximate minimizers of convex functionals. In addition, we give convergence rate estimates for a sequence approximating the minimum value of such a functional.

Keywords Proximal point algorithm · prox-Tikhonov algorithm · Monotone operator · Control conditions · Strong convergence · Convex function · Minimizer · Minimum value

Mathematics Subject Classification (2000) 47J25 · 47H05 · 47H09

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A map $T : H \to H$ is said to be nonexpansive if for every $x, y \in H$ the inequality $\|Tx - Ty\| \le \|x - y\|$ holds. In the case when $\|Tx - Ty\| \le a\|x - y\|$ holds for some $a \in (0, 1)$, then T is said to be a contraction with Lipschitz constant a. We recall that a mapping $A : D(A) \subset H \to 2^H$ is said to be a monotone operator if

$$\langle x - x', y - y' \rangle \ge 0, \quad \forall\, (x, y), (x', y') \in A.$$

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In other words, the graph of A, $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$ is a monotone subset of $H \times H$. The operator A is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone operator. It is well known that if A is maximal monotone and $\beta > 0$, then the resolvent of A, the operator $J_\beta : H \to H$ defined by $J_\beta(x) = (I + \beta A)^{-1}(x)$, is single-valued and nonexpansive (see, e.g. [13]).

For a fixed $u \in H$ and $t \in (0, 1)$, let z_t denote the fixed point of the contraction T_t given by the rule $x \mapsto tu + (1 - t)Tx$, i.e.,

$$z_t = tu + (1 - t)Tz_t. \tag{1}$$

The strong convergence of z_t to a fixed point of T was proved in 1967 by Browder [3]. This result of Browder has been widely used in the theory of fixed points and extended in different directions by several authors. Motivated by Browder's (implicit) convergence result, Halpern [7] considered the (explicit) iteration

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \text{for any } u, x_0 \in H \text{ with } \alpha_n \in (0, 1) \text{ and all } n \geq 0, \tag{2}$$

in a Hilbert space and proved that under certain assumptions on α_n, the sequence $\{x_n\}$ given by the iterative process (2) is strongly convergent, and the limit is the point of $F(T) = \{x \in H \mid Tx = x\}$ which is nearest to u. Later, Lions [12] proved the strong convergence of (2) still in a Hilbert space under the control conditions

$$\text{(C1)} \lim_{n\to\infty} \alpha_n = 0, \quad \text{(C2)} \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \text{(C3)} \lim_{n\to\infty} \frac{(\alpha_{n+1} - \alpha_n)}{\alpha_{n+1}^2} = 0.$$

Unfortunately, Lions' result excludes the natural choice $\alpha_n = n^{-1}$. This was overcome in 1992 by Wittmann [17] who showed strong convergence of $\{x_n\}$ under the control conditions (C1), (C2), and

$$\text{(C4)} \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

In 2002, Xu [18] studied algorithm 2 extensively. First, he showed that in a Banach space setting, $\{x_n\}$ still maintains its strong convergence on removing the square in the denominator of (C3), thereby improving Lions' result twofold. The conditions used were (C1), (C2), and

$$\text{(C5)} \lim_{n\to\infty} \frac{(\alpha_{n+1} - \alpha_n)}{\alpha_{n+1}} = 0, \quad \text{or equivalently,} \quad \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1.$$

He then showed that the conditions (C3) and (C4) are not comparable, and did the same for (C4) and (C5). Xu then observed that Halpern actually showed that the conditions (C1) and (C2) are necessary to have strong convergence to the metric projection of u on $F(T)$. This provided a partial answer to Reich's question: Concerning $\{\alpha_n\}$, what are the necessary and sufficient conditions for $\{x_n\}$ to converge strongly? To the best of our knowledge, the other part of the question concerning sufficiency remains open. However, in a recent paper of Suzuki [16], it is shown that if the nonexpansive mapping T in (2) is of the form $T := \lambda S + (1 - \lambda)I$ (with $\lambda \in (0, 1)$, S a nonexpansive mapping and I the identity operator), then the conditions (C1) and (C2) are not only necessary for $\{x_n\}$ to converge strongly, but they are also sufficient. In fact, Suzuki showed strong convergence of the iterative process

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda Sx_n + (1 - \lambda)x_n), \quad \text{for any } u, x_0 \in H \text{ and all } n \geq 0, \tag{3}$$

in Banach spaces. The same result was obtained by Chidume and Chidume [4] independently. Very recently, He et. al. [8] showed also in Banach spaces that if the nonexpansive map S

above is replaced by the resolvent, J_{β_n}, of an m-accretive operator, then strong convergence is still guaranteed under (C1), (C2), and the condition

$$\text{(C6)} \quad \lim_{n\to\infty} (\beta_{n+1} - \beta_n) = 0,$$

with β_n bounded from below away from zero.

Notice that it is possible to prove strong convergence results if one replaces the nonexpansive map T in algorithm 2 by a sequence of nonexpansive mappings. For instance, one may consider the iterative process, known as the (modified) proximal point algorithm of Halpern-type, defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\beta_n} x_n, \quad \text{for any } u, x_0 \in H \text{ and all } n \geq 0, \tag{4}$$

where $\{\beta_n\} \subset (0, \infty)$. Under additional assumptions on β_n, the strong convergence of $\{x_n\}$ defined by (4) can be obtained. In 2000, Kamimura and Takahashi [10] showed that $\{x_n\}$ is strongly convergent to the point of the set $F(J_c) = \{x \in H : J_c x = x\} = A^{-1}(0)$ (for all $c > 0$) nearest to u if one assumes (C1), (C2) and $\beta_n \to \infty$. In fact, they considered the following algorithm which is the inexact form of algorithm 4:

$$\begin{cases} y_n \approx J_{\beta_n} x_n, & \text{for all } n \geq 0, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \end{cases} \tag{5}$$

for any $u, x_0 \in H$, where the criterion for the approximate computation of y_n is given by

$$\|y_n - J_{\beta_n} x_n\| \leq \delta_n \quad \text{with} \quad \sum_{n=0}^{\infty} \delta_n < \infty.$$

It is worth mentioning that Xu [18] also obtained the same result independently, and in [1] (see also [2]), we extended this result to include non-summable errors, e_n. It is unclear if the same conclusion can also be derived for bounded β_n and the general condition that the error sequence tend to zero in norm. We refer the interested reader to the paper of Rockafellar [14] to see what happens in the case when $\alpha_n = 0$ for all $n \geq 0$.

The so called prox-Tikhonov regularization method have also been under investigation from several researchers. In 2006, Xu [19] extended the result of Lehdili and Moudafi [11] by considering the iterative process

$$x_{n+1} = J_{\beta_n}(\alpha_n u + (1 - \alpha_n) x_n + e_n), \quad \text{for any } u, x_0 \in H \text{ and all } n \geq 0, \tag{6}$$

were $\{e_n\}$ is a sequence of errors, and proved strong convergence of $\{x_n\}$ defined by (6) to the metric projection of u into the fixed point set $A^{-1}(0)$ under the control conditions which appear as a combination of α_n and β_n. More precisely, his conditions were

$$\text{(C7)} \sum_{n=0}^{\infty} \left| 1 - \frac{\alpha_n \beta_{n+1}}{\alpha_{n+1} \beta_n} \right| < \infty \quad \text{or,} \quad \text{(C8)} \lim_{n\to\infty} \frac{1}{\alpha_n} \left(1 - \frac{\alpha_n \beta_{n+1}}{\alpha_{n+1} \beta_n} \right) = 0.$$

Note that for $\beta_n \to \infty$, the natural choices of $\alpha_n = n^{-1}$ and $\beta_n = n$, fails under both conditions. In fact, for any choice of α_n and β_n, condition (C7) is impossible to achieve as shall be shown in this paper, (see Remark 4). In another result of Xu, Theorem 3.3 [19], it is shown that for summable errors, strong convergence is still maintained under the conditions (C1), (C2), (C4), and β_n bounded (from above and from below away from zero) with (C9) (as defined below) being satisfied. Song and Yang [15] established strong convergence of the prox-Tikhonov algorithm 6 when the errors are summable, (C1), (C2), (C4) being satisfied, and the following condition on β_n imposed: β_n is bounded from below away from zero with either

$$\text{(C9)} \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \quad \text{or} \quad \text{(C9)}' \sum_{n=0}^{\infty} \frac{|\beta_{n+1} - \beta_n|}{\beta_{n+1}} < \infty.$$

They remarked that their result (Theorem 2) contains Theorem 3.3 [19] as a special case. Although this seems to be the case at first glance, it turns out that the two theorems are equivalent. In fact, the condition (C9)$'$ on β_n is equivalent to (C9) and β_n bounded from below away from zero. Obviously, from this equivalence follows the equivalence of the two theorems. This equivalence is not so obvious and it is discussed in Lemma 4 below.

The main purpose of this paper is to prove strong convergence of $\{x_n\}$ conforming to the iterative process

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\beta_n} x_n + e_n, \quad \text{for any } u, x_0 \in H \text{ and all } n \geq 0, \tag{7}$$

under new conditions on α_n and β_n. The conditions we are about to introduce will allow choices such as $\alpha_n = n^{-1}$ and $\beta_n = n$, and they are weaker than those previously studied, so our results can be viewed as significant improvements and refinements of previously known results. Theorem 5 deals with the conditions

$$\text{either} \quad \text{(C10)} \sum_{n=1}^{\infty} \left| \frac{\alpha_{n-1}}{\beta_n} - \frac{\alpha_n}{\beta_{n+1}} \right| < \infty \quad \text{or,} \quad \text{(C11)} \lim_{n\to\infty} \frac{1}{\alpha_n \beta_n^2} (\alpha_{n-1}\beta_{n+1} - \alpha_n \beta_n) = 0,$$

and Theorem 6 is concerned with the conditions

$$\text{(C6)}^* \lim_{n\to\infty} \left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right) = 0, \quad \text{and either}$$

$$\text{(C12)} \lim_{n\to\infty} \frac{(\alpha_n - \alpha_{n-1})}{\alpha_{n-1}\beta_n} = 0 \quad \text{or,} \quad \text{(C13)} \sum_{n=1}^{\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} < \infty.$$

In particular, our results provide an answer to the question we asked in [2]: Can one design a proximal point algorithm by choosing appropriate regularization parameters α_n such that strong convergence of $\{x_n\}$ is preserved, for $\|e_n\| \to 0$ and β_n bounded? Of course, for constant β_n, (C10) reduces to (C4) and (C11) reduces to (C5).

If A is the subdifferential of a proper, convex, lower semicontinuous function $\varphi : H \to (-\infty, \infty]$, then our convergence results provide sequences which converge strongly to the minimum point of φ nearest to u. In addition, we give convergence rate estimates for a sequence converging to $\inf \varphi$ (see Theorem 7 of Sect. 4). The reader interested in theoretical and practical aspects of convex and non-convex optimization theory is referred to the recent excellent six-volume resource [6]. See also [5,9].

2 Preliminaries

In the sequel, H is a real Hilbert space, F denotes the set $A^{-1}(0) = \{x \in H : J_c x = x\} = F(J_c)$ for all $c > 0$, and given any sequence $\{x_n\}$, its weak ω-limit set will be denoted by $\omega_w(\{x_n\})$, that is,

$$\omega_w(\{x_n\}) := \left\{ x \in H \mid x_{n_k} \rightharpoonup x \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \right\}.$$

Here "$\rightharpoonup$" denotes weak convergence. Setting

$$v_n := \frac{x_n - \alpha_{n-1}u - e_{n-1}}{1 - \alpha_{n-1}}, \tag{8}$$

we see that (7) can be reformulated as

$$v_{n+1} = J_{\beta_n}(\alpha_{n-1}u + (1 - \alpha_{n-1})v_n + e_{n-1}), \quad \text{for } n \geq 1. \tag{9}$$

It is worth pointing out that, for $\alpha_n \to 0$ and $e_n \to 0$, the algorithms (7) and (9) are equivalent, that is, $\{v_n\}$ converges if and only if $\{x_n\}$ does. We shall therefore always use either form of the algorithm at our convenience. Obviously, (9) has the form (6), with α_{n-1}, β_n, and e_{n-1} instead of α_n, β_n, and e_n. If we consider (9) instead of (6), then conditions (C7) and (C8) take the form

$$\text{(C7)}' \sum_{n=1}^{\infty} \left| 1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n \beta_n} \right| < \infty \quad \text{or,} \quad \text{(C8)}' \lim_{n\to\infty} \frac{1}{\alpha_{n-1}} \left(1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n \beta_n} \right) = 0.$$

Theorem 4 and Remark 4 are concerned with these conditions. Let us now recall some Lemmas which will be useful in proving our main results. The first Lemma can be proved easily.

Lemma 1 *For all $x, y \in H$, we have*

$$\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y\rangle.$$

Lemma 2 (Resolvent Identity). *For any $\beta, \gamma > 0$, and $x \in H$, the identity*

$$J_\beta x = J_\gamma \left(\frac{\gamma}{\beta} x + \left(1 - \frac{\gamma}{\beta} \right) J_\beta x \right)$$

holds true.

Proof The proof of this Lemma is well known, but we provide it for the sake of completeness. Let $\beta, \gamma > 0$, and $x \in H$ be arbitrary but fixed. Set $y := J_\beta x$. Then using the definition of the resolvent, we have

$$y = J_\beta x \;\Leftrightarrow\; y + \beta A y \ni x \;\Leftrightarrow\; y + \gamma A y \ni \frac{\gamma}{\beta} x + \left(1 - \frac{\gamma}{\beta} \right) y \;\Leftrightarrow\; y$$
$$= J_\gamma \left(\frac{\gamma}{\beta} x + \left(1 - \frac{\gamma}{\beta} \right) y \right).$$

This completes the proof of the resolvent identity. □

Lemma 3 [18]. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n + c_n, \quad n \geq 0,$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ satisfy the conditions: (i) $\{a_n\} \subset [0, 1]$, with $\sum_{n=0}^{\infty} a_n = \infty$, (ii) $\limsup_{n\to\infty} b_n \leq 0$, and (iii) $c_n \geq 0$ for all $n \geq 0$ with $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.

We next show that any sequence of positive real numbers satisfying the condition of (C9)$'$ is bounded (with the lower bound being strictly positive).

Lemma 4 *For any sequence $\{b_n\}$ of positive real numbers, the following conditions are equivalent:* (i) $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$ *and* $0 < \liminf_{n\to\infty} b_n$ $(=\lim_{n\to\infty} b_n)$, (ii) $\sum_{n=0}^{\infty} \frac{|b_{n+1} - b_n|}{b_n} < \infty$, *and* (iii) $\sum_{n=0}^{\infty} \frac{|b_{n+1} - b_n|}{b_{n+1}} < \infty$.

Proof First, it is easily seen that (i) $\Rightarrow$ (ii), and (i) $\Rightarrow$ (iii). Now let us prove that (ii) $\Rightarrow$ (i). For this, it suffices to show that there exist constants $m, M > 0$ such that $m \leq b_n \leq M$ for all $n = 0, 1, \ldots$

From (ii), there exists a sequence $\{a_n\} \subset \mathbb{R}$, such that $\sum_{n=0}^{\infty} |a_n| < \infty$, and

$$\frac{b_{n+1} - b_n}{b_n} = a_n \Leftrightarrow \frac{b_{n+1}}{b_n} = 1 + a_n, \quad n = 0, 1, \ldots$$

Note that in particular, $\lim_{n\to\infty} a_n = 0$. Therefore, we may assume without any loss of generality that $|a_n| < 1$ for all n. Then by simple induction, we have

$$\frac{b_n}{b_0} = \prod_{k=0}^{n-1}(1 + a_k). \tag{10}$$

Since $1 + x \leq \exp(x)$ for all $x \geq 0$, it follows from (10) that

$$\frac{b_n}{b_0} = \prod_{k=0}^{n-1}(1 + a_k) \leq \prod_{k=0}^{n-1}(1 + |a_k|) \leq \exp\left(\sum_{k=0}^{n-1} |a_k|\right) \leq \exp\left(\sum_{k=0}^{\infty} |a_k|\right) =: M_0 < \infty. \tag{11}$$

On the other hand,

$$\sum_{k=0}^{\infty} |a_k| < \infty \Leftrightarrow \prod_{k=0}^{\infty}(1 - |a_k|) > 0,$$

and again from (10) we obtain

$$\frac{b_n}{b_0} = \prod_{k=0}^{n-1}(1 + a_k) \geq \prod_{k=0}^{n-1}(1 - |a_k|) \geq \prod_{k=0}^{\infty}(1 - |a_k|) =: m_0 > 0. \tag{12}$$

The conclusion then follows from (11) and (12). Replacing b_n by b_n^{-1} in (ii), one readily gets (iii), showing that (iii) $\Rightarrow$ (i) as desired. □

Let the mapping $h : H \to H$ be defined by $x \mapsto tu + (1-t)J_c x + e(t)$ for $c > 0, u \in H$ and $t \in (0, 1)$, where $e = e(t)$ is a given function defined on $(0, 1)$ with values in H. For any fixed t (and c, u), one can easily check that the map h is a contraction with Lipschitz constant $1 - t$. The Banach contraction principle asserts that h has a unique fixed point, say, z_t. That is,

$$z_t = tu + (1 - t)J_c z_t + e(t) \quad \text{for} \quad c > 0 \text{ and } u \in H. \tag{13}$$

In fact z_t depends on u and c as well.

Theorem 1 *Take any $c > 0$ and $u \in H$, and assume*

$$t^{-1}\|e(t)\| \to 0 \quad \text{as } t \to 0^+. \tag{14}$$

If $F \neq \emptyset$, then $\{z_t\}$ defined in (13) converges strongly as $t \to 0^+$ to the point of F nearest to u, denoted by $P_F u$. Moreover, this limit is attained uniformly with respect to $c \geq \delta$ for every $\delta > 0$.

Proof For every $p \in F$, we have from Lemma 1

$$\|z_t - p\|^2 \leq (1 - t)^2\|z_t - p\|^2 + 2t\langle u - p + t^{-1}e(t), z_t - p\rangle.$$

In other words,

$$(2-t)\|z_t - p\|^2 \le 2\langle u - p + t^{-1}e(t), z_t - p\rangle. \tag{15}$$

This shows that $\{z_t\}$ is bounded as $t \to 0^+$. Now setting

$$v_t := (1-t)^{-1}(z_t - tu - e(t)) = J_c z_t,$$

we see that $\{v_t\}$ is also bounded as $t \to 0^+$ and the weak ω-limit sets of $\{z_t\}$ and $\{v_t\}$ (as $t \to 0^+$) coincide, that is, $\omega_w(\{z_t\}) = \omega_w(\{v_t\})$. Since

$$Av_t \ni \frac{1}{c}(z_t - v_t) \to 0 \quad \text{as } t \to 0^+,$$

we have $\omega_w(\{z_t\}) \subset F$. By (14) and (15) with $p = P_F u$ we get

$$\limsup_{t\to 0^+} \|z_t - P_F u\|^2 \le 0,$$

which shows that

$$\lim_{t\to 0^+} \|z_t - P_F u\| = 0.$$

Obviously, the above limit is attained uniformly with respect to $c \ge \delta$ for every $\delta > 0$. □

Remark 1 Theorem 1 is an extension of Theorem 3.1 in [19], since v_t converges strongly to $P_F u$ (as $t \to 0^+$) if and only if z_t does. We note that Theorem 3.1 in [19] contains a mistake, since the strong limit of v_t (as $t \to 0^+$) is not attained uniformly for $c > 0$ (but for $c \ge \delta$ for every $\delta > 0$).

3 Main results

We devote this section to demonstrate the strong convergence of algorithm 7 under different sets of assumptions on the parameters α_n and β_n. We begin by proving a strong convergence result satisfying similar conditions to those of Lions. One of the conditions

$$(\text{C3})' \ \lim_{n\to\infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2} = 0,$$

is weaker than Lions' condition (C3) in the case when α_n is decreasing.

Theorem 2 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \ne \emptyset$. For any fixed $u, x_0 \in H$, let $\{x_n\}$ be the sequence generated by algorithm (7) with the conditions: (i) $\alpha_n \in (0,1)$, (C1), (C2) and (C3)$'$, (ii) either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\|e_n\|/\alpha_n \to 0$, and (iii) $\beta_n \in (0,\infty)$ with (C6)$'$ $\lim_{n\to\infty} \beta_n = \beta$ for some $\beta > 0$, being satisfied. Then $\{x_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof Note that it was shown in [19] that $\{x_n\}$ is bounded if $\sum_{n=0}^{\infty} \|e_n\| < \infty$. Also it was shown in [2] that $\{x_n\}$ is bounded if $\{e_n/\alpha_n\}$ is bounded. For each n, let z_n be the unique fixed point of the contraction $x \mapsto \alpha_n u + (1-\alpha_n)J_\beta x$. According to Theorem 1, $z_n \to P_F u$ as $n \to \infty$. Therefore it is enough to show that $\|x_n - z_n\| \to 0$ as $n \to \infty$. For this purpose, we estimate $\|x_{n+1} - z_{n+1}\|$ as follows

$$\|x_{n+1} - z_{n+1}\| \le \|x_{n+1} - z_n\| + \|z_n - z_{n+1}\|. \tag{16}$$

Noting that $z_n = \alpha_n u + (1-\alpha_n) J_\beta z_n$ and the fact that J_β is nonexpansive for all $\beta > 0$, we get

$$\begin{aligned}\|x_{n+1} - z_n\| &\le (1-\alpha_n)\|J_{\beta_n} x_n - J_\beta z_n\| + \|e_n\| \\ &\le (1-\alpha_n)\|J_{\beta_n} x_n - J_{\beta_n} z_n\| + \|J_{\beta_n} z_n - J_\beta z_n\| + \|e_n\| \\ &\le (1-\alpha_n)\|x_n - z_n\| + \frac{|\beta-\beta_n|}{\beta}\|z_n - J_\beta z_n\| + \|e_n\| \\ &\le (1-\alpha_n)\|x_n - z_n\| + \alpha_n \frac{|\beta-\beta_n|}{\beta}\|u - J_\beta z_n\| + \|e_n\|, \end{aligned} \tag{17}$$

where the third inequality follows from the application of the resolvent identity. On the other hand, we compare z_n and z_{n+1} as follows

$$\begin{aligned}\|z_n - z_{n+1}\| &= \|(\alpha_n - \alpha_{n+1})(u - J_\beta z_{n+1}) + (1-\alpha_n)(J_\beta z_n - J_\beta z_{n+1})\| \\ &\le |\alpha_n - \alpha_{n+1}|\|u - J_\beta z_{n+1}\| + (1-\alpha_n)\|z_n - z_{n+1}\|,\end{aligned}$$

which gives

$$\|z_n - z_{n+1}\| \le \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} K, \tag{18}$$

where K is a positive constant such that $\|u - J_\beta z_n\| \le K$ for all n. Combining (16), (17) and (18) we get

$$\|x_{n+1} - z_{n+1}\| \le (1-\alpha_n)\|x_n - z_n\| + \alpha_n b_n + c_n,$$

where

$$b_n = K\left[\frac{|\beta-\beta_n|}{\beta} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2}\right] \to 0 \quad \text{and} \quad c_n = \|e_n\| \quad \text{with} \quad \sum_{n=0}^{\infty}\|e_n\| < \infty,$$

or

$$\|x_{n+1} - z_{n+1}\| \le (1-\alpha_n)\|x_n - z_n\| + \alpha_n b_n',$$

where

$$b_n' = K\left[\frac{|\beta-\beta_n|}{\beta} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2}\right] + \frac{\|e_n\|}{\alpha_n} \to 0$$

for the case $\|e_n\|/\alpha_n \to 0$. In either case Lemma 3 gives the required conclusion. □

Remark 2 For $\beta > 0$ and $\beta_n = \beta + (-1)^n/(n+1)$, the condition (C6)′ is satisfied, whereas (C9) is not, showing that our condition on β_n is weaker than the one used in the following theorem due to Xu [19]. On the other hand, the sequences $\alpha_n = n^{-3/4}$ and $\alpha_n = 1/\ln n$ satisfy condition (i) of Theorem 2. Since (C3) and (C3)′ are not comparable to (C4) (see Remark 3.1 [18]), Theorem 2 is new.

Theorem 3 [19]. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let $\{x_n\}$ be the sequence generated by algorithm (7) with the conditions: (i) $\alpha_n \in (0,1)$, (C1), (C2) and (C4), (ii) $\beta_n \in (0,\infty)$ with $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$ and $0 < \liminf_{n\to\infty}\beta_n$ $(= \lim_{n\to\infty}\beta_n)$, being satisfied. If $\sum_{n=0}^{\infty}\|e_n\| < \infty$, then $\{x_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Remark 3 Although it appears from Lemma 3 and inequality (18) that

$$\sum_{n=0}^{\infty} \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} < \infty$$

can be a possible assumption on α_n, there is no sequence $\{\alpha_n\} \subset (0, 1)$ satisfying (C1) and this condition. Indeed, if this condition is satisfied, then Lemma 4 implies that α_n is bounded below away from zero, contradicting (C1).

We next give a result similar to Theorem 3.2 of Xu [19]. In the next result, if we consider algorithm 6 instead of algorithm 7, then we can prove the same result with (C8)′ being replaced by (C8). In that case, the result extends Theorem 3.2 [19] to a larger class of errors which include those that are non-summable and still converge to zero in norm. Moreover, we can show that Theorem 3.2 [19] fails to hold under the condition (C7).

Theorem 4 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let $\{x_n\}$ be the sequence generated by algorithm (7), where (i) $\alpha_n \in (0, 1)$, with (C1), and (C2), (ii) either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\|e_n\|/\alpha_n \to 0$, and (iii) $\beta_n \in (0, \infty)$ with $\liminf_{n\to\infty} \beta_n > 0$, $\beta_{n+1} \geq \alpha_n \beta_n$ and $(C8)'$. Then $\{x_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof For each fixed n, let y_n be the unique fixed point of the contraction $x \mapsto \alpha_{n-1}u + (1-\alpha_{n-1})J_{\beta_n}x$. Then according to Theorem 1, $y_n \to P_F u$ as $n \to \infty$. Set

$$v_n := \frac{x_n - \alpha_{n-1}u - e_{n-1}}{1-\alpha_{n-1}} \quad \text{and} \quad w_n := \frac{y_n - \alpha_{n-1}u}{1-\alpha_{n-1}}. \tag{19}$$

As a consequence of the boundedness of $\{x_n\}$ and $\{y_n\}$ (see [2] and [19]), the sequences $\{v_n\}$ and $\{w_n\}$ are bounded. Also by virtue of (19), $w_n \to P_F u$ as $n \to \infty$. It follows from (7) and the definition of y_n that

$$v_{n+1} = J_{\beta_n}((1-\alpha_{n-1})v_n + \alpha_{n-1}u + e_{n-1}) \quad \text{and} \quad w_n = J_{\beta_n}((1-\alpha_{n-1})w_n + \alpha_{n-1}u).$$

As before, using the nonexpansivity of the resolvent, we estimate $\|v_{n+1} - w_{n+1}\|$ as follows

$$\begin{aligned}\|v_{n+1} - w_{n+1}\| &\leq \|v_{n+1} - w_n\| + \|w_{n+1} - w_n\| \\ &\leq (1-\alpha_{n-1})\|v_n - w_n\| + \|w_{n+1} - w_n\| + \|e_{n-1}\|. \end{aligned} \tag{20}$$

Now using the resolvent identity and the nonexpansivity of the resolvent, we can estimate $\|w_{n+1} - w_n\|$ as follows

$$\begin{aligned}\|w_{n+1} - w_n\| &= \left\| J_{\beta_n}\left(\frac{\beta_n}{\beta_{n+1}}((1-\alpha_n)w_{n+1} + \alpha_n u) + \left(1 - \frac{\beta_n}{\beta_{n+1}}\right) w_{n+1}\right)\right. \\ &\qquad \left. - J_{\beta_n}((1-\alpha_{n-1})w_n + \alpha_{n-1}u)\right\| \\ &\leq \left(1 - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)\|w_{n+1} - w_n\| + \left|\alpha_{n-1} - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right| K,\end{aligned}$$

which gives

$$\|w_{n+1} - w_n\| \leq \left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right| K, \tag{21}$$

for some positive constant K. Combining (20) and (21) we get

$$\|v_{n+1} - w_{n+1}\| \le (1 - \alpha_{n-1})\|v_n - w_n\| + \left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n \beta_n}\right| K + \|e_{n-1}\|. \quad (22)$$

Hence from Lemma 3, we see that $\|v_n - w_n\| \to 0$, and the proof is complete. □

Remark 4 In view of Lemma 3 and (22), it is tempting to infer that the theorem is still valid under the condition (C7)′. However we show that this condition is impossible to attain for any sequences $\{\beta_n\}$ and $\{\alpha_n\}$ satisfying the conditions of the above theorem. To this end, we assume that (C7)′ holds true. Denote $b_n := \alpha_{n-1}/\beta_n$. Then

$$\sum_{n=1}^{\infty}\left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n \beta_n}\right| < \infty \Leftrightarrow \sum_{n=1}^{\infty} \frac{|b_{n+1} - b_n|}{b_{n+1}} < \infty.$$

Therefore, it follows from Lemma 4 that

$$\liminf_{n\to\infty} \frac{\alpha_{n-1}}{\beta_n} = \liminf_{n\to\infty} b_n > 0,$$

which implies that $\beta_n \to 0$ (since $\alpha_n \to 0$). This is a contradiction as β_n is bounded below away from zero.

However, if we allow $\beta_n \to 0$, then Theorem 1 is no longer applicable. Indeed, from $w_n = J_{\beta_n}((1 - \alpha_{n-1})w_n + \alpha_{n-1}u)$, we have

$$\frac{\alpha_{n-1}}{\beta_n}(u - w_n) \in A w_n. \quad (23)$$

From the above inclusion relation, we can not derive $\omega_w(\{w_n\}) \subset F := A^{-1}(0)$, even if w_n is strongly convergent (since by (21), $\sum_{n=1}^{\infty} \|w_{n+1} - w_n\| < \infty$) because α_{n-1}/β_n may not necessarily converge to zero. Therefore, in this case $\{x_n\}$ is still strongly convergent (according to (22)) but we can not derive that its limit is in F. In fact, its limit need not be in F. We give an example to that effect.

Example 1 Let $\beta_n = 1/n$ and $\alpha_n = 1/(n+2)$ for $n \ge 1$. Then we have

$$1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n \beta_n} = \frac{1}{(n+1)^2} =: a_n, \quad \text{for all } n \ge 1, \quad \text{and} \quad \frac{\beta_{n+1}}{\alpha_n} \to 1 \quad \text{as } n \to \infty.$$

Clearly the condition $\beta_{n+1} \ge \alpha_n \beta_n$ for all $n \ge 1$ is fulfilled. Let $H = \mathbb{R}$, and let the sequence $\{e_n\} \subset \mathbb{R}$ satisfy either the condition $\sum_{n=0}^{\infty} |e_n| < \infty$ or $|e_n|/\alpha_n \to 0$, (for example, $|e_n| = (n+2)^{-2}$ or $|e_n| = 1/(n \ln n)$ for $n \ge 2$ with $\sum_{n=2}^{\infty} |e_n| = \infty$, respectively), and let $A : D(A) = [0, \infty) \subset \mathbb{R} \to \mathbb{R}$ be defined by

$$Ax = \begin{cases} ax, & \text{if } x > 0, \\ (-\infty, 0], & \text{if } x = 0, \\ \emptyset, & \text{if } x < 0, \end{cases}$$

for some $a > 0$. Then if $u > 0$, we have for sufficiently large n, $\alpha_{n-1}u + e_{n-1} > 0$ and

$$\begin{aligned} 0 < w_n &= J_{\beta_n}((1 - \alpha_{n-1})w_n + \alpha_{n-1}u + e_{n-1}) \\ &= \frac{1}{1 + \beta_n a}((1 - \alpha_{n-1})w_n + \alpha_{n-1}u + e_{n-1}), \end{aligned}$$

which implies that $w_n \to w_\infty := \frac{1}{1+a}u \notin F = \{0\}$. Hence $x_n \to w_\infty \notin F$. The same conclusion is true if $u < 0$.

The argument given above shows that if β_n is bounded away from zero in Theorem 3.2 of [19], then the condition (C7) is impossible to achieve. Also the above example shows that the result may not hold if $\beta_n \to 0$.

We now give an example to show the applicability of Theorem 4.

Example 2 Choose $\beta_n = \beta_0 > 0$ for all n, $\alpha_n = (n+1)^{-1/2}$ and $\|e_n\| = 1/(n+1)$ for all $n \geq 0$.

Theorem 5 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let $\{x_n\}$ be the sequence generated by algorithm (7) where conditions (i) and (ii) of Theorem 4 are fulfilled. If $\beta_n \in (0, \infty)$ is increasing and either (C10) or (C11) is satisfied, then $\{x_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof We know that $\{x_n\}$ (and hence $\{v_n\}$) is bounded, see Theorem 2.
Claim: $\limsup_{n\to\infty}\langle u - P_F u, x_n - P_F u\rangle \leq 0$.
Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging weakly to some x_∞, such that

$$\limsup_{n\to\infty}\langle u - P_F u, x_n - P_F u\rangle = \lim_{k\to\infty}\langle u - P_F u, x_{n_k} - P_F u\rangle = \langle u - P_F u, x_\infty - P_F u\rangle.$$

To prove the claim, we only need to show that $x_\infty \in F$, or more generally $\omega_w(\{x_n\}) \subset F$. If β_n is unbounded, then the conclusion follows from the inclusion relation

$$\frac{v_{n+1} - v_n}{\beta_n} + A(v_{n+1}) \ni \frac{\alpha_{n-1}}{\beta_n}(u - v_n) + \frac{1}{\beta_n} e_{n-1}. \tag{24}$$

Otherwise, from Eq. 9, the boundedness of $\{\|e_n\|/\alpha_n\}$ and $\{v_n\}$, the nonexpansivity of J_{β_n} and taking advantage of the resolvent identity, we can compare v_{n+2} and v_{n+1} as follows

$$\begin{aligned}
\|v_{n+2} - v_{n+1}\| &= \Big\| J_{\beta_n}\left(\tfrac{\beta_n}{\beta_{n+1}}((1-\alpha_n)v_{n+1} + \alpha_n u + e_n) + \left(1 - \tfrac{\beta_n}{\beta_{n+1}}\right) v_{n+2}\right) \\
&\quad - J_{\beta_n}((1-\alpha_{n-1})v_n + \alpha_{n-1}u + e_{n-1})\Big\| \\
&\leq \Big\| \left(1 - \tfrac{\beta_n}{\beta_{n+1}}\right)(v_{n+2} - v_{n+1}) + \left(1 - \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\right)(v_{n+1} - v_n) \\
&\quad + \left(\alpha_{n-1} - \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\right)\left(v_n - u - \tfrac{e_{n-1}}{\alpha_{n-1}}\right) + \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\left(\tfrac{e_n}{\alpha_n} - \tfrac{e_{n-1}}{\alpha_{n-1}}\right)\Big\| \\
&\leq \left(1 - \tfrac{\beta_n}{\beta_{n+1}}\right)\|v_{n+2} - v_{n+1}\| + \left(1 - \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\right)\|v_{n+1} - v_n\| \\
&\quad + \left|\alpha_{n-1} - \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\right| K + \frac{\alpha_n\beta_n}{\beta_{n+1}}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{\|v_{n+2} - v_{n+1}\|}{\beta_{n+1}} &\leq \left(1 - \frac{\alpha_n\beta_n}{\beta_{n+1}}\right)\frac{\|v_{n+1} - v_n\|}{\beta_n} + \left|\frac{\alpha_{n-1}}{\beta_n} - \frac{\alpha_n}{\beta_{n+1}}\right| K + \frac{\alpha_n}{\beta_{n+1}}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\| \\
&\leq \left(1 - \frac{\alpha_n\beta_0}{\beta_{n+1}}\right)\frac{\|v_{n+1} - v_n\|}{\beta_n} + \left|\frac{\alpha_{n-1}}{\beta_n} - \frac{\alpha_n}{\beta_{n+1}}\right| K + \frac{\alpha_n}{\beta_{n+1}}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|.
\end{aligned}$$

Similarly, for the case $\sum_{n=0}^{\infty} \|e_n\| < \infty$, we have

$$\begin{aligned}\|v_{n+2} - v_{n+1}\| &\le \left\| \left(1 - \tfrac{\beta_n}{\beta_{n+1}}\right)(v_{n+2} - v_{n+1}) + \left(1 - \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\right)(v_{n+1} - v_n) \right. \\ &\quad \left. + \left(\alpha_{n-1} - \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\right)(v_n - u) + \left(\tfrac{\beta_n}{\beta_{n+1}} e_n - e_{n-1}\right) \right\| \\ &\le \left(1 - \tfrac{\beta_n}{\beta_{n+1}}\right)\|v_{n+2} - v_{n+1}\| + \left(1 - \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\right)\|v_{n+1} - v_n\| \\ &\quad + \left|\alpha_{n-1} - \tfrac{\alpha_n\beta_n}{\beta_{n+1}}\right| K' + \left\| \tfrac{\beta_n}{\beta_{n+1}} e_n - e_{n-1} \right\|,\end{aligned}$$

which implies that

$$\frac{\|v_{n+2} - v_{n+1}\|}{\beta_{n+1}} \le \left(1 - \frac{\alpha_n\beta_0}{\beta_{n+1}}\right)\frac{\|v_{n+1} - v_n\|}{\beta_n} + \left|\frac{\alpha_{n-1}}{\beta_n} - \frac{\alpha_n}{\beta_{n+1}}\right| K' + \frac{1}{\beta_0}\left(\|e_n\| + \|e_{n-1}\|\right).$$

Denote $a_n := \alpha_n\beta_0/\beta_{n+1}$. Since $\{\alpha_n\}$ satisfy $\alpha_n \in (0, 1)$, $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, so do $\{a_n\}$. Therefore, from Lemma 3, we have (in both cases)

$$\frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0 \Leftrightarrow \|v_{n+1} - v_n\| \to 0.$$

Moreover, (24) implies that $\omega_w(\{v_n\}) \subset F$, and from (8), we derive $\omega_w(\{v_n\}) = \omega_w(\{x_n\})$, hence the claim.

Finally we show that $\{x_n\}$ converges strongly to $P_F u$. We have from Lemma 1

$$\|x_{n+1} - P_F u\|^2 \le (1 - \alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{n+1} - P_F u \right\rangle. \quad (25)$$

In the case when $\|e_n\|/\alpha_n \to 0$, inequality (25) implies by Lemma 3 that $x_n \to P_F u$. If $\sum_{n=0}^{\infty} \|e_n\| < \infty$, then we derive from inequality (25)

$$\|x_{n+1} - P_F u\|^2 \le (1 - \alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \langle u - P_F u, x_{n+1} - P_F u\rangle + K\|e_n\|,$$

for some $K > 0$, and Lemma 3 again implies that $x_n \to P_F u$ as desired. □

Remark 5 The condition (C10) is weaker than the conditions (C4) and (C9) if $\beta_n \ge \delta$ for all n and for some $\delta > 0$. Indeed,

$$\begin{aligned}\left|\frac{\alpha_{n-1}}{\beta_n} - \frac{\alpha_n}{\beta_{n+1}}\right| &\le \frac{1}{\beta_n}|\alpha_{n-1} - \alpha_n| + \alpha_n \left|\frac{1}{\beta_n} - \frac{1}{\beta_{n+1}}\right| \\ &\le \frac{1}{\delta}\left[|\alpha_{n-1} - \alpha_n| + \frac{|\beta_{n+1} - \beta_n|}{\delta}\right].\end{aligned}$$

Note that if $\beta_n = n^2$ for $n \ge 1$, then (C10) holds true for any choice of $\alpha_n \in (0, 1)$.

Remark 6 Observe that (C11) is satisfied for $\beta_n = n$ and $\alpha_n = (n + 1)^{-1}$, whereas the condition (C8)′ of Theorem 3 fails. Moreover, (C11) works if β_n is constant and α_n taken as before but (C8)′ fails.

Although the condition (C10) is weaker than (C4) and (C9) if $\liminf_{n\to\infty} \beta_n > 0$, our result is restricted only to those β_n's which are increasing. The next result is designed to cater for those β_n's who does not satisfy this restrictive condition. It is actually an extension and improvement of Theorem 3 above. Our proof differs from those given in [15] and [19], and it relies on the equivalence of the algorithms 6 and 7. Note that it was observed in [15] that a gap exists in the proof of Theorem 3. We remark here that our method of transforming

Eq. 9 into Eq. 7 is an alternative way of solving this gap as can be seen from the proof of Theorem 6 below.

Theorem 6 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let the sequence $\{x_n\}$ be generated by algorithm (7) with the following conditions being satisfied:* (i) $\alpha_n \in (0, 1)$, *(C1), (C2),* (ii) *either* $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *or* $\|e_n\|/\alpha_n \to 0$, (iii) $\liminf_{n\to\infty} \beta_n > 0$, *and (C6)*. If either (C12) or (C13) hold, then $\{x_n\}$ (and hence $\{v_n\}$) converges strongly to $P_F u$, the projection of u on F.*

Proof We know from [2] and [19] that $\{x_n\}$ is bounded. For $\|e_n\|/\alpha_n \to 0$, we have, (by the resolvent identity and the nonexpansivity of the resolvent),

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le (1-\alpha_{n-1}) \left\| J_{\beta_n} x_n - J_{\beta_n}\left(\tfrac{\beta_n}{\beta_{n-1}} x_{n-1} + \left(1 - \tfrac{\beta_n}{\beta_{n-1}}\right) J_{\beta_{n-1}} x_{n-1}\right)\right\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \cdot \|u - J_{\beta_n} x_n + e_n/\alpha_n\| + \alpha_{n-1}\|e_n/\alpha_n - e_{n-1}/\alpha_{n-1}\| \\
&\le (1-\alpha_{n-1}) \left\| \tfrac{\beta_n}{\beta_{n-1}} (x_n - x_{n-1}) + \left(1 - \tfrac{\beta_n}{\beta_{n-1}}\right)(x_n - J_{\beta_{n-1}} x_{n-1})\right\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \cdot \|u - J_{\beta_n} x_n + e_n/\alpha_n\| + \alpha_{n-1}\|e_n/\alpha_n - e_{n-1}/\alpha_{n-1}\| \\
&\le (1-\alpha_{n-1}) \tfrac{\beta_n}{\beta_{n-1}} \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|u - J_{\beta_n} x_n + e_n/\alpha_n\| \\
&\quad + \alpha_{n-1}\left|1 - \tfrac{\beta_n}{\beta_{n-1}}\right| \left\| u - J_{\beta_{n-1}} x_{n-1} + \frac{e_{n-1}}{\alpha_{n-1}}\right\| + \alpha_{n-1}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\|,
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\beta_n} &\le (1-\alpha_{n-1}) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_{n-1}\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| K + \frac{\alpha_{n-1}}{\beta_n}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\| \\
&\quad + K\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n},
\end{aligned} \tag{26}$$

for some positive constant K. From Lemma 3 and inequality (26), we have

$$\frac{\|x_{n+1} - x_n\|}{\beta_n} \to 0,$$

which is equivalent to

$$\frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0.$$

Hence we can derive (see (24) above), $\omega_w(\{x_n\}) = \omega_w(\{v_n\}) \subset F$. Consequently, we have

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle \le 0. \tag{27}$$

Note that for some positive constant C, $|\beta_{n+1}^{-1} - \beta_n^{-1}| \le C$ (since $\liminf_{n\to\infty} \beta_n > 0$) and

$$x_{n+1} - x_n = (\alpha_n - \alpha_{n-1})(u - J_{\beta_n} x_n) + (e_n - e_{n-1}) + (1-\alpha_{n-1})(J_{\beta_n} x_n - J_{\beta_{n-1}} x_{n-1}),$$

so that in the case when $\sum_{n=1}^{\infty} \|e_n\| < \infty$, we again get inequality (27) on applying similar arguments as above.

As in the proof of Theorem 5, we derive strong convergence of $\{x_n\}$ to $P_F u$. □

Remark 7 Clearly (C6)* is weaker than the conditions

$$\text{(C14)} \sum_{n=1}^{\infty} \left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| < \infty \quad \text{and} \quad \text{(C15)} \lim_{n\to\infty} \frac{1}{\alpha_n}\left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right) = 0,$$

both of which hold true if $\alpha_n = n^{-1}$ and $\beta_n = n$ while (C6) fails for this choice of β_n. However, both (C6) and (C6)* hold if $\beta_n = \ln n$. We point out that the first inequality in Remark 5 suggest that for $\liminf_{n\to\infty} \beta_n > 0$, the condition (C10) is weaker than (C4) and (C14). Also, the condition (C14) is weaker than (C9) whenever $\liminf_{n\to\infty} \beta_n > 0$ holds. But the condition that β_n is increasing is stronger than the assumption $\liminf_{n\to\infty} \beta_n > 0$, so there are cases in which the following corollary is applicable and Theorem 5 is not. We remark that both (C4) and (C5) are not satisfied by

$$\alpha_n = \begin{cases} 1/n, & \text{if } n \text{ is odd,} \\ 1/(2n), & \text{if } n \text{ is even.} \end{cases}$$

This choice of α_n however fulfills the assumptions of (C13) for the case when $\beta_n = n$ and (C12) for any $\beta_n \to \infty$.

Corollary 1 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let the sequence $\{x_n\}$ be generated by algorithm (7), where $\alpha_n \in (0, 1)$ and $\beta_n \in (0, \infty)$, with the conditions* (i) *and* (ii) *taken as in Theorem 6, and $\liminf_{n\to\infty} \beta_n > 0$ with either (C14) or (C15). If either (C12) or (C13) hold, then $\{x_n\}$ (and hence $\{v_n\}$) converges strongly to $P_F u$.*

The following corollary is an extension of Theorem 3.

Corollary 2 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. For any fixed $u, x_0 \in H$, let the sequence $\{x_n\}$ be generated by algorithm 7, where $\alpha_n \in (0, 1)$ and $\beta_n \in (0, \infty)$, with the conditions (i) and (ii) taken as in Theorem 6, and (iii) $\liminf_{n\to\infty} \beta_n > 0$ and either (C9) or (C16) $\lim_{n\to\infty} \frac{1}{\alpha_n}\left(1 - \frac{\beta_n}{\beta_{n+1}}\right) = 0$. If either (C12) or (C13) hold, then $\{x_n\}$ (hence $\{v_n\}$) converges strongly to $P_F u$.*

We give an example to show that the conditions of (iii) are different.

Example 3 Let $\alpha_n = (n+2)^{-1/4}$ and $\beta_n = 2(n+1)(n+2)^{-1}$ for all $n \geq 0$. Then α_n and β_n satisfy both conditions of *(iii)* while $\beta_n = (n+1)$ and α_n as above satisfy only (C16).

Remark 8 Let us observe that if $\|e_n\|/\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \|e_n\| = \infty$, then automatically $\sum_{n=0}^{\infty} \alpha_n = \infty$. Also the trend that has been followed by many authors in order to obtain strong convergence of the PPA was to use the criterion which restricts the error sequence to be summable. We have deviated from this tradition by allowing any sequence of errors converging strongly to zero and still derived strong convergence of the PPA. Indeed, if $\sum_{n=0}^{\infty} \|e_n\| = \infty$ and $\|e_n\| \to 0$, then we can construct (or choose) a sequence $\{\alpha_n\}$ of parameters depending on $\{e_n\}$ such that the condition $\|e_n\|/\alpha_n \to 0$ holds (for example $\alpha_n = \sqrt{\|e_n\|}$ if $e_n \neq 0$ and all n big enough). Otherwise, (i.e., if $\sum_{n=0}^{\infty} \|e_n\| < \infty$), we can choose freely (independent of e_n) $\alpha_n \in (0, 1)$ such that the conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ are satisfied.

4 The case when A is a subdifferential

Recall that the subdifferential of a proper and convex function $\varphi : H \to (-\infty, +\infty]$ is the operator (possibly multivalued) $\partial\varphi : H \to H$ defined by

$$\partial\varphi(x) = \{w \in H \mid \varphi(x) - \varphi(v) \leq \langle w, x - v\rangle, \quad \forall\, v \in H\}.$$

If in addition, φ is lower semicontinuous, then its subdifferential is a maximal monotone operator and a point $p \in H$ minimizes φ if and only if $0 \in \partial\varphi(p)$. In other words, $A^{-1}(0)$ for $A = \partial\varphi$ is the set of minimum points of φ. Note that for $A = \partial\varphi$, where φ is a proper, convex and lower semicontinuous function, algorithm (9) is equivalent to

$$v_{n+1} = \arg\min_{x\in H}\varphi_n(x),$$

where

$$\varphi_n(x) = \varphi(x) + \frac{1}{2\beta_n}\|x - \alpha_{n-1}u - (1-\alpha_{n-1})v_n - e_{n-1}\|^2.$$

Obviously, φ_n is a coercive function having a unique minimizer v_{n+1} due to the quadratic term added to $\varphi(x)$.

Under the assumptions of the previously proved results, $\{v_n\}$ (equivalently, $\{x_n\}$) converges strongly to the minimizer of φ nearest to u.

We now give two convergence rate estimates for the residual $\varphi(w_{k+1}) - \varphi(z)$ where φ is a proper, convex and lower semicontinuous function and z is an arbitrary point of H, and

$$w_n = \sigma_n^{-1}\sum_{k=1}^{n}\beta_k v_{k+1} \quad \text{where } \sigma_n = \sum_{k=1}^{n}\beta_k. \tag{28}$$

In general, if a sequence $\{v_n\}$ converges strongly (resp. weakly) to a point, say p, then the sequence of its weighted means with positive weights $\{\beta_k\}$ defined by (28) also converges strongly (resp. weakly) to the same limit p, provided $\sigma_n \to \infty$.

Theorem 7 *Let $A = \partial\varphi$ and $A^{-1}(0) \neq \emptyset$ where $\varphi : H \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function. For any fixed $u, v_1 \in H$, let $\{v_n\}$ be the sequence generated by algorithm (9) and $\{w_n\}$ be as in (28).*

- *If $\sum_{k=1}^{\infty}\|e_{k-1}\| < \infty$, then for some $K > 0$, the following estimate holds*

$$\varphi(w_n) - \varphi(z) \leq \frac{\|v_1 - z\|^2 + K\left(\sum_{k=1}^{n}\alpha_{k-1} + \sum_{k=1}^{n}\|e_{k-1}\|\right)}{2\sigma_n}, \quad \text{for all } z \in H. \tag{29}$$

- *If $\{e_n/\alpha_n\}$ is bounded, then for some $M > 0$, we have*

$$\varphi(w_n) - \varphi(z) \leq \frac{\|v_1 - z\|^2 + M\sum_{k=1}^{n}\alpha_{k-1}}{2\sigma_n}, \quad \text{for all } z \in H. \tag{30}$$

If in addition, $\sigma_n^{-1}\sum_{k=1}^{n}\alpha_{k-1} \to 0$ as $n \to \infty$, then $\varphi(w_n) \to \inf_{y\in H}\varphi(y)$.

Proof Let us prove estimate (30). Note that for $A = \partial\varphi$, we have from (9),

$$\alpha_{k-1}(u - v_k) + e_{k-1} + (v_k - v_{k+1}) \in \beta_k\partial\varphi(v_{k+1}),$$

and for all $z \in H$, we have from the boundedness of $\{e_k/\alpha_k\}$ and $\{v_k\}$ (see the proof of Theorem 1 [2]),

$$\begin{aligned} 2\beta_k(\varphi(v_{k+1}) - \varphi(z)) &\leq 2\langle v_k - v_{k+1}, v_{k+1} - z\rangle + 2\alpha_{k-1}\langle u - v_k + e_{k-1}/\alpha_{k-1}, v_{k+1} - z\rangle \\ &\leq \left(\|v_k - z\|^2 - \|v_{k+1} - v_k\|^2 - \|v_{k+1} - z\|^2\right) + M\alpha_{k-1}, \end{aligned} \tag{31}$$

for some $M > 0$. Summing (31) from $k = 1, \ldots, n$ and rearranging terms, we get

$$2\varphi(z) + \frac{\|v_1 - z\|^2 + M\sum_{k=1}^{n}\alpha_{k-1}}{\sigma_n} \geq 2\frac{\sum_{k=1}^{n}\beta_k\varphi(v_{k+1})}{\sigma_n} \geq 2\varphi\left(\frac{\sum_{k=1}^{n}\beta_k v_{k+1}}{\sigma_n}\right). \tag{32}$$

Therefore (30) follows from (32). The proof of the other estimate is similar. The final assertion of the theorem is obvious. □

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